Local distribution of the parts of unequal partitions in arithmetic progressions II

Cécile Dartyge, Mihály Szalay

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Cécile Dartyge (Nancy) and Mihály Szalay (Budapest) *

1. Introduction

This paper contains the main parts of the proofs of the results announced in [6]. We recall below some notations and the main result of this paper but we recommend to the reader to study first [6]. Let $d \in \mathbb{N}^*$, $\mathcal{D}$ a non-empty subset of $\{1, \dots, d\}$ and $\mathcal{D}^c = \{1, \dots, d\} \smallsetminus \mathcal{D}$ its complement. Let $\mathcal{R}_{\mathcal{D}} = \{N_r : r \in \mathcal{D}\}$ be a multiset of $|\mathcal{D}|$ non-negative integers. The main goal of our work is to obtain an asymptotic formula for $\Pi_d^*(n, \mathcal{R}_{\mathcal{D}})$, the number of unequal partitions of n with exactly N_r parts congruent to r modulo d for all $r \in \mathcal{D}$. We adopt the convention $\Pi_d^*(0, \mathcal{R}_{\mathcal{D}}) = 1$ if $\mathcal{R}_{\mathcal{D}} = \{0, ..., 0\}$ and 0 otherwise.

Recall that if $n \geqslant 1$ and $\Pi_d^*(n, \mathcal{R}_{\mathcal{D}}) \geqslant 1$ then n satisfies

$$n \equiv R_{\mathcal{D}} \pmod{\delta}, \tag{1·1}$$

where $R_{\mathcal{D}} = \sum_{r \in \mathcal{D}} r N_r$ and δ is the *g. c. d.* of the elements of $\mathcal{D}^c \cup \{d\}$. In the introduction of [6], we observed that the N_r, $r \in \mathcal{D}$ may be expected to be close to k_0 with

$$k_0 := \frac{2\sqrt{3}\log 2}{\pi} \frac{\sqrt{n}}{d}. \tag{1·2}$$

More precisely we suppose that for all $r \in \mathcal{D}$ we have

$$|N_r - k_0| \leqslant \frac{n^{\frac{1}{4}}\sqrt{\log n}}{d^{1/3}|\mathcal{D}|^{2/3} w(n)}, \tag{1·3}$$

where $w(n)$ is a non-decreasing function such that $w(n) \to \infty$ if $n \to \infty$. Let us recall the main result of [6]

Theorem 1.1. *Let $\varepsilon > 0$. The following two propositions hold.*

(i) Let $d \leqslant n^{1/4-\varepsilon}$, $\mathcal{D} = \{1, \dots, d\}$ and $n \equiv R_{\mathcal{D}} \pmod{d}$. Let $\mathcal{R} = \mathcal{R}_{\mathcal{D}} = \{N_1, \dots, N_d\}$ be a multiset of integers satisfying (1·3). *Then we have*

$$\Pi_d^*(n, \mathcal{R}_D) = (1 + o(1)) q(n) \frac{d}{\sqrt{1 - \frac{12(\log 2)^2}{\pi^2}}} \Big(\frac{d}{2\sqrt{3n}}\Big)^{d/2}$$
$$\times \exp\Big\{ - \frac{2\sqrt{3}\log^2 2}{\pi\big(1 - \frac{12(\log 2)^2}{\pi^2}\big)\sqrt{n}} \Big(\sum_{r=1}^{d} (N_r - k_0)\Big)^2 - \frac{\pi d}{2\sqrt{3n}} \sum_{r=1}^{d} (N_r - k_0)^2 \Big\}.$$

(ii) We suppose now that $d \leqslant n^{1/6-\varepsilon}$ and $\mathcal{D} \subset \{1, \dots, d\}$. Then under (1·1) *and* (1·3) *we have*

$$\Pi_d^*(n, \mathcal{R}_D) = q(n) \frac{\delta(1 + o(1))}{\sqrt{1 - \frac{12|\mathcal{D}|(\log 2)^2}{d\pi^2}}} \Big(\frac{d}{2\sqrt{3n}}\Big)^{|\mathcal{D}|/2}$$
$$\times \exp\Big(- \frac{2\sqrt{3}(\log 2)^2}{\pi\Big(1 - \frac{12|\mathcal{D}|(\log 2)^2}{d\pi^2}\Big)\sqrt{n}} \Big(\sum_{r \in \mathcal{D}} (N_r - k_0)\Big)^2 - \frac{\pi d}{2\sqrt{3n}} \sum_{r \in \mathcal{D}} (N_r - k_0)^2 \Big).$$

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First we complete the proof of Theorem 1.1 in the case $\mathcal{D} = \{1, \ldots, d\}$, after we will handle the complementary case when $\mathcal{D}^c \neq \varnothing$. The last sections are devoted to the proofs of the different corollaries of [6].

2. The term S_2

We begin to assume that

$$d \leqslant n^{\frac{1}{2}-\varepsilon} \tag{2·1}$$

with some fixed positive ε and

$$|k - k_0| = o\Big(\frac{\sqrt{n}}{d}\Big). \tag{2·2}$$

Let

$$x_0 := \frac{\pi}{2\sqrt{3n}}, \quad t := dx_0. \tag{2·3}$$

Then

$$k_0 t = k_0 d x_0 = \log 2. \tag{2·4}$$

We also suppose that

$$|N_r - k_0| = o\Big(\frac{\sqrt{n}}{d}\Big) \qquad (r = 1, \ldots, d). \tag{2·5}$$

In Section 4 of [6] we proved that as $n \to \infty$ then we have

$$\prod_{r=1}^{d} g_{N_r}(dx_0) = \exp\Big(\frac{\pi^2}{12x_0} + \frac{(\log 2)^2}{2x_0} + o(\sqrt{n})\Big) \tag{2·6}$$

According to the notations of [6] Sections 3 and 4, we have

$$|S_2| \leqslant \frac{d}{2\pi} \int_{3\pi x_0 \leqslant |y| \leqslant \pi/d} \Big\{ \prod_{r=1}^{d} |g_{N_r}(d(x_0 + iy))| \Big\} \exp((n - R - Q)x_0) \, \mathrm{d}y.$$

The main part of this section is the following lemma

Lemma 2.1. *Under the notations and hypotheses* (1·2), (2·1), (2·2), (2·3), (2·4), *and* $3\pi x_0 \leqslant |y| \leqslant \pi/d$, *we have*

$$|g_k(d(x_0 + iy))| \leqslant g_k(dx_0) \exp\Big(\frac{-1 + o(1)}{4dx_0}\Big).$$

Proof. This time we start out from the first expression of g_k and develop the logarithms:

(2·7)

$$g_k(w) = \exp\Big(\sum_{\nu=1}^{k} \log \frac{1}{1 - \exp(-\nu w)}\Big) = \exp\Big(\sum_{\nu=1}^{k} \sum_{m=1}^{\infty} \frac{1}{m} \exp(-\nu m w)\Big)$$

$$= \exp\Big(\sum_{m=1}^{\infty} \frac{1}{m} \sum_{\nu=1}^{k} \exp(-\nu m w)\Big) = \exp\Big(\sum_{\nu=1}^{k} \exp(-\nu w) + \sum_{m=2}^{\infty} \frac{1}{m} \sum_{\nu=1}^{k} \exp(-\nu m w)\Big).$$

We take the moduli

$$
\begin{aligned}
|g_k(w)| &\leqslant \exp\Big(\Big|\sum_{\nu=1}^{k}\exp(-\nu w)\Big| + \sum_{m=2}^{\infty}\frac{1}{m}\sum_{\nu=1}^{k}\exp(-\nu m t)\Big)\\
&= g_k(t)\exp\Big(\Big|\sum_{\nu=1}^{k}\exp(-\nu w)\Big| - \sum_{\nu=1}^{k}\exp(-\nu t)\Big)\\
&= g_k(t)\exp\Big(\frac{|1-\exp(-kw)|}{|\exp(w)-1|} - \frac{1-\exp(-kt)}{\exp(t)-1}\Big)\\
&\leqslant g_k(t)\exp\Big(\frac{1+\exp(-kt)}{|\exp(w)-1|} - \frac{1-\exp(-kt)}{\exp(t)-1}\Big).
\end{aligned}
$$

When $|\Im m\, w| \leqslant \pi$,

$$
|\exp(w)-1|^2 = (\exp(t)-1)^2 + 4\mathrm{e}^t(\sin(b/2))^2 \geqslant 4(\sin(b/2))^2 \geqslant \frac{4|b|^2}{\pi^2},
$$

thus

$$
|g_k(w)| \leqslant g_k(t)\exp\Big(\frac{1+\exp(-kt)}{\frac{2}{\pi}|\Im m\, w|} - \frac{1-\exp(-kt)}{\exp(t)-1}\Big)
$$

if $|\Im m\, w| \leqslant \pi$. Therefore, $3\pi x_0 \leqslant |y| \leqslant \pi/d$ implies that

$$
\begin{aligned}
|g_k(d(x_0+iy))| &\leqslant g_k(dx_0)\exp\Big(\frac{1+\exp(-kdx_0)}{\frac{2}{\pi}d|y|} - \frac{1-\exp(-kdx_0)}{\exp(dx_0)-1}\Big)\\
&\leqslant g_k(dx_0)\exp\Big(\frac{1+\exp(-kdx_0)}{6dx_0} - \frac{1-\exp(-kdx_0)}{\exp(dx_0)-1}\Big).
\end{aligned}
$$

By (1·2), (2·1), (2·2), (2·3), (2·4),

$$
\begin{aligned}
|g_k(d(x_0+iy))| &\leqslant g_k(dx_0)\exp\Big(\frac{\frac{3}{2}+o(1)}{6dx_0} - \frac{\frac{1}{2}+o(1)}{\exp(dx_0)-1}\Big)\\
&= g_k(dx_0)\exp\Big(\frac{\frac{3}{2}+o(1)}{6dx_0} - \frac{\frac{1}{2}+o(1)}{dx_0} + O(1)\Big)\\
&= g_k(dx_0)\exp\Big(-\frac{1+o(1)}{4dx_0}\Big).
\end{aligned}
$$

This ends the proof of Lemma 2.1. □

By (2·5) we obtain for S_2,

$$
\begin{aligned}
|S_2| &\leqslant \Big\{\prod_{r=1}^{d} g_{N_r}(dx_0)\Big\}\exp\Big(-\frac{1+o(1)}{4x_0}\Big)\exp((n-R-Q)x_0)\\
&= \exp\Big(\frac{\pi\sqrt{n}}{\sqrt{3}} - \frac{\sqrt{3n}}{2\pi} + o(\sqrt{n})\Big),
\end{aligned}
\tag{2·8}
$$

by (2·6) and according to the estimates of Q and R obtained in Sections 2 and 4 of [6].

3. The term S_1

Next, we will try to give a similar and simple estimation for $n^{-\frac{5}{8}+\varepsilon} \leqslant |y| \leqslant 3\pi x_0$.

Lemma 3.1. *(i) Under the notations and hypotheses* (1·2), (2·1), (2·2), (2·3), (2·4), *for* $n^{-\frac{5}{8}+\varepsilon} \leqslant |y| \leqslant 3\pi x_0$ *we have*

$$|g_k(d(x_0+iy))| \leqslant g_k(dx_0)\exp\Big(-\frac{\sqrt{3}n^{1/4+2\varepsilon}}{27\pi^5 d}\Big).$$

(ii) Under the notations and hypotheses (1·2), (2·1), (2·2), (2·3), (2·4), *for* $n^{-\frac{3}{4}+\frac{\varepsilon}{3}} \leqslant |y| \leqslant n^{-\frac{5}{8}+\varepsilon}$ *we have*

$$|g_k(d(x_0+iy))| \leqslant g_k(dx_0)\exp\Big(-\frac{\sqrt{3}n^{2\varepsilon/3}}{27\pi^5 d}\Big).$$

Proof of (i). We suppose that $n^{-\frac{5}{8}+\varepsilon} \leqslant |y| \leqslant 3\pi x_0$. By (2·7) we have

$$g_k(w) = \exp\Big(\sum_{\nu=1}^{k}\exp(-\nu w) + \sum_{m=2}^{\infty}\frac{1}{m}\sum_{\nu=1}^{k}\exp(-\nu m w)\Big).$$

We study again $|g_k(w)|$:

$$\begin{aligned}
|g_k(w)| &\leqslant \exp\Big(\Re e\Big(\sum_{\nu=1}^{k}\exp(-\nu w)\Big) + \sum_{m=2}^{\infty}\frac{1}{m}\exp(-\nu m t)\Big)\\
&= g_k(t)\exp\Big(\sum_{\nu=1}^{k}\Re e\,(\exp(-\nu w)) - \sum_{\nu=1}^{k}\exp(-\nu t)\Big)\\
&= g_k(t)\exp\Big(\sum_{\nu=1}^{k}\exp(-\nu t)(\Re e\ \exp(-\nu i b)-1)\Big)\\
&= g_k(t)\exp\Big(\sum_{\nu=1}^{k}\exp(-\nu t)(\cos(\nu b)-1)\Big)\\
&= g_k(t)\exp\Big(-2\sum_{\nu=1}^{k}\exp(-\nu t)\sin^2\big(\frac{\nu|b|}{2}\big)\Big).
\end{aligned}$$

Let $K_0 := \lfloor \frac{k_0}{3\log 2}\rfloor$. If $k = k_0 + o\big(\frac{\sqrt{n}}{d}\big)$ then $k > K_0$ for n large enough :

$$\begin{aligned}
|g_k(w)| &\leqslant g_k(t)\exp\Big(-2\sum_{\nu=1}^{K_0}\exp(-\nu t)\sin^2\big(\frac{\nu|b|}{2}\big)\Big)\\
&\leqslant g_k(t)\exp\Big(-2\sum_{\nu=1}^{K_0}\exp(-k_0 t)\sin^2\big(\frac{\nu|b|}{2}\big)\Big) = g_k(t)\exp\Big(-\sum_{\nu=1}^{K_0}\sin^2\Big(\frac{\nu|b|}{2}\Big)\Big).
\end{aligned}$$

Writing $w = d(x_0+iy)$ and using the inequality $|\sin t| \geqslant \frac{2|t|}{\pi}$ for $|t| \leqslant \pi/2$, we obtain

$$|g_k(d(x_0+iy))| \leqslant g_k(dx_0)\exp\Big(-\sum_{\nu=1}^{K_0}\Big(\frac{\nu d|y|}{\pi}\Big)^2\Big)$$

since

$$\frac{\nu d|y|}{2} \leqslant \frac{K_0 d 3\pi x_0}{2} \leqslant \frac{k_0 d\pi x_0}{2\log 2} = \frac{\pi}{2}.$$

Since $\sum_{\nu=1}^{K_0} \nu^2 \geqslant \frac{K_0^3}{3}$, we have

$$|g_k(d(x_0+iy))| \leqslant g_k(dx_0)\exp\Big(-\big(\frac{d|y|}{\pi}\big)^2\frac{K_0^3}{3}\Big) \leqslant g_k(dx_0)\exp\Big(-\frac{\sqrt{3}n^{1/4+2\varepsilon}}{\pi^5 3^3 d}\Big).$$

(ii) can be obtained similarly. □

By (2·5),

$$\begin{aligned}
&\Big|\frac{d}{2\pi}\int_{n^{-\frac{5}{8}+\varepsilon}\leqslant|y|\leqslant 3\pi x_0}\Big\{\prod_{r=1}^{d} g_{N_r}(d(x_0+iy))\Big\}\exp((n-R-Q)(x_0+iy))\,\mathrm{d}y\Big|\\
&\leqslant \frac{d}{2\pi}\int_{n^{-\frac{5}{8}+\varepsilon}\leqslant|y|\leqslant 3\pi x_0}\Big|\prod_{r=1}^{d} g_{N_r}(d(x_0+iy))\Big|\exp((n-R-Q)x_0)\,\mathrm{d}y\\
&\leqslant \Big\{\prod_{r=1}^{d} g_{N_r}(dx_0)\Big\}\exp\Big(-\frac{\sqrt{3}n^{\frac{1}{4}+2\varepsilon}}{\pi^5 3^3}\Big)\exp((n-R-Q)x_0).
\end{aligned}$$

We have to stop here since the previously error term $o(\sqrt{n})$ is rough. Otherwise the above proof can be applied, *e. g.*, for $n^{-\frac{3}{4}+\frac{\varepsilon}{3}} \leqslant |y| \leqslant n^{-\frac{5}{8}+\varepsilon}$ and results that

$$\begin{aligned}
&\left|\frac{d}{2\pi}\int_{n^{-\frac{3}{4}+\frac{\varepsilon}{3}}\leqslant|y|\leqslant n^{-\frac{5}{8}+\varepsilon}}\left\{\prod_{r=1}^{d} g_{N_r}(d(x_0+iy))\right\}\exp((n-R-Q)(x_0+iy))\,\mathrm{d}y\right|\\
&\leqslant \left\{\prod_{r=1}^{d} g_{N_r}(dx_0)\right\}\exp\Big(-\frac{\sqrt{3}n^{2\varepsilon/3}}{\pi^5 3^3}\Big)\exp((n-R-Q)x_0).
\end{aligned}$$

Finally we obtain for S_1:

$$|S_1| \ll \Big\{\prod_{r=1}^{d} g_{N_r}(dx_0)\Big\}\exp\Big(-\frac{\sqrt{3}n^{2\varepsilon/3}}{\pi^5 3^3}\Big)\exp((n-R-Q)x_0). \tag{3·1}$$

4. The function g_k in the range $|y| < y_1$.

Let $|y| \leqslant y_1 = n^{-\frac{3}{4}+\frac{\varepsilon}{3}}$, $w = t+ib = dx_0 + idy$. Now $\frac{|b|}{t} = O(n^{-\frac{1}{4}+\frac{\varepsilon}{3}})$.
In this section we work with a general subset $\mathcal{D} \subset \{1,\dots,d\}$.
Instead of (2·2) and (2·5), we suppose that

$$|k-k_0| \leqslant \frac{n^{\frac{1}{4}}\sqrt{\log n}}{d^{1/3}|\mathcal{D}|^{2/3}w(n)} \quad\text{and}\quad |N_r-k_0| \leqslant \frac{n^{\frac{1}{4}}\sqrt{\log n}}{d^{1/3}|\mathcal{D}|^{2/3}w(n)} \quad (r\in\mathcal{D}) \tag{4·1}$$

where $w(n)$ is a non-decreasing function such that $w(n) \to \infty$ if $n \to \infty$. The aim of this paragraph is to obtain an asymptotic formula for $g_k(x_0+iy)$ for $|y| \leqslant y_1$. Instead of (2·1), we suppose that

$$d \leqslant n^{\frac{1}{4}-2\varepsilon}. \tag{4·2}$$

Thus (4·1) implies (2·2) and (2·5). We will prove the following Lemma :

Lemma 4.1. *Under* (4·1) *we have*

$$g_k(w) = f(w)\exp\Big\{ -\frac{C_2}{t} + (k-k_0)\log 2 - \frac{(k-k_0)^2 t}{2} + \frac{\log 2}{2} \\ + ib\Big(\frac{C_2}{t^2} + \frac{k_0\log 2}{t} - k_0(k-k_0)\Big) + b^2\Big(\frac{C_2}{t^3} + \frac{k_0\log 2}{t^2} + \frac{k_0^2}{2t}\Big) + o\Big(\frac{n^{-\varepsilon}}{|\mathcal{D}|}\Big)\Big\}.$$

We have again by Lemma 4.1 of [6]

$$g_k(w) = f(w)\exp\Big\{ -\frac{1}{w}\sum_{m=1}^{\infty}\frac{1}{m^2}\exp(-kmw) + \frac{1}{2}\sum_{m=1}^{\infty}\frac{1}{m}\exp(-kmw) + O(k^{-1})\Big\}.$$

By (1·2) and (2·2),

$$\frac{1}{k} = O\big(\frac{d}{\sqrt{n}}\big) = O\big(\frac{1}{d}\frac{d^2}{\sqrt{n}}\big).$$

Then $\frac{1}{k} = O(n^{-2\varepsilon}/d)$ and $\frac{|b|}{t} = o(n^{-\varepsilon}d^{-1})$. Since now $|y| \leqslant y_1$, it is possible to replace w by t in the different $\exp(-kmw)$, in cost of an admissible error term:

Lemma 4.2. *(i) We have*

$$\sum_{m=1}^{\infty}\frac{1}{m}\exp(-kmw) = \sum_{m=1}^{\infty}\frac{1}{m}\exp(-kmt) + O\big(\frac{|b|}{t}\big).$$

(ii) We have

$$\frac{1}{w}\sum_{m=1}^{\infty}\frac{1}{m^2}\exp(-kmw) = \frac{1}{w}\sum_{m=1}^{\infty}\frac{1}{m^2}\exp(-kmt) - \frac{kib}{w}\sum_{m=1}^{\infty}\frac{1}{m}\exp(-kmt) \\ - \frac{k^2b^2}{2w}\sum_{m=1}^{\infty}\exp(-kmt) + O\Big(\frac{k^3|b|^3}{|w|}\Big)\sum_{m=1}^{\infty} m\exp(-kmt).$$

Proof. By standard approximations we have

$$\begin{aligned}\sum_{m=1}^{\infty}\frac{1}{m}\exp(-kmw) &= \sum_{m=1}^{\infty}\frac{1}{m}(1-(1-\exp(-kmib)))\exp(-kmt) \\ &= \sum_{m=1}^{\infty}\frac{1}{m}\exp(-kmt) + \sum_{m=1}^{\infty}\frac{1}{m}\exp(-kmt)O(km|b|) \\ &= \sum_{m=1}^{\infty}\frac{1}{m}\exp(-kmt) + O\big(\frac{k|b|}{e^{kt}-1}\big) = \sum_{m=1}^{\infty}\frac{1}{m}\exp(-kmt) + O\big(\frac{|b|}{t}\big).\end{aligned}$$

This proves (i). Next we prove (ii). We have

$$\begin{aligned}\frac{1}{w}\sum_{m=1}^{\infty}\frac{1}{m^2}\exp(-kmw) &= \frac{1}{w}\sum_{m=1}^{\infty}\frac{1}{m^2}\exp(-kmt)\exp(-ikmb) \\ &= \frac{1}{w}\sum_{m=1}^{\infty}\frac{1}{m^2}\exp(-kmt)\big\{1 - ikmb - \frac{(kmb)^2}{2} + O((km|b|)^3)\big\}.\end{aligned}$$

It remains to develop to end the proof of Lemma 4.2. □

We also have

$$\sum_{m=1}^{\infty} m\exp(-kmt) = \frac{\exp(-kt)}{(1-\exp(-kt))^2} = \frac{1}{(\exp(kt)-1)(1-\exp(-kt))} \leqslant \frac{1}{(kt)^2},$$

since for $u>0$,

$$\mathrm{e}^u - 1 = u\sum_{n=0}^{\infty}\frac{u^n}{(n+1)!} > u\sum_{n=0}^{\infty}\frac{u^n}{n!2^n} = u\mathrm{e}^{\frac{u}{2}},$$

thus $(1-\mathrm{e}^{-u}) > u\mathrm{e}^{-u/2}$ and $(\mathrm{e}^u-1)(1-\mathrm{e}^{-u}) > u^2$.

This gives for $g_k(w)$:

$$\begin{aligned} g_k(w) = f(w)\exp\Big\{ &-\frac{1}{w}\sum_{m=1}^{\infty}\frac{1}{m^2}\exp(-kmt) \\ &+\frac{ikb}{w}\sum_{m=1}^{\infty}\frac{1}{m}\exp(-kmt) + \frac{k^2b^2}{2w}\sum_{m=1}^{\infty}\exp(-kmt) \\ &+\frac{1}{2}\sum_{m=1}^{\infty}\frac{1}{m}\exp(-kmt) + o\Big(\frac{n^{-\varepsilon}}{d}\Big) + O\Big(k\big(\frac{|b|}{t}\big)^3\Big)\Big\}. \end{aligned}$$

The next step of the proof of Lemma 4.1 consists of "replacing" $\frac{1}{w}$ by $\frac{1}{t}$ and computing the terms arisen by this manipulation. We use the formula

$$\frac{1}{w} = \frac{1}{t(1-(-i\frac{b}{t}))} = \frac{1}{t}\Big(1 - i\frac{b}{t} - \frac{b^2}{t^2} + O\Big(\frac{|b|^3}{t^3}\Big)\Big). \tag{4·3}$$

This gives for $g_k(w)$

$$\begin{aligned} g_k(w) = f(w)\exp\Big\{ &-\frac{1}{t}\sum_{m=1}^{\infty}\frac{1}{m^2}\exp(-kmt) + \frac{ib}{t^2}\sum_{m=1}^{\infty}\frac{1}{m^2}\exp(-kmt) \\ &+\frac{b^2}{t^3}\sum_{m=1}^{\infty}\frac{1}{m^2}\exp(-kmt) + O\Big(\frac{|b|^3}{t^4}\Big) + \frac{ikb}{t}\sum_{m=1}^{\infty}\frac{1}{m}\exp(-kmt) \\ &+\frac{kb^2}{t^2}\sum_{m=1}^{\infty}\frac{1}{m}\exp(-kmt) + O\Big(\frac{k|b|^3}{t^3}\log\Big(1+\frac{1}{\mathrm{e}^{kt}-1}\Big)\Big) \\ &+\frac{k^2b^2}{2t}\sum_{m=1}^{\infty}\exp(-kmt) + O\Big(\frac{k^2|b|^3}{t^2(e^{kt}-1)}\Big) \\ &+\frac{1}{2}\sum_{m=1}^{\infty}\frac{1}{m}\exp(-kmt) + o\Big(\frac{n^{-\varepsilon}}{d}\Big) + O\Big(\frac{k|b|^3}{t^3}\Big)\Big\}. \end{aligned}$$

We collect the terms with ib, the terms with b^2:

$$\begin{aligned} g_k(w) = f(w)\exp\Big\{ &-\frac{1}{t}\sum_{m=1}^{\infty}\frac{1}{m^2}\exp(-kmt) + \frac{1}{2}\sum_{m=1}^{\infty}\frac{1}{m}\exp(-kmt) \\ &+ib\Big(\frac{1}{t^2}\sum_{m=1}^{\infty}\frac{1}{m^2}\exp(-kmt) + \frac{k}{t}\sum_{m=1}^{\infty}\frac{1}{m}\exp(-kmt)\Big) \\ &+b^2\Big(\frac{1}{t^3}\sum_{m=1}^{\infty}\frac{1}{m^2}\exp(-kmt) + \frac{k}{t^2}\sum_{m=1}^{\infty}\frac{1}{m}\exp(-kmt) + \frac{k^2}{2t}\sum_{m=1}^{\infty}\exp(-kmt)\Big) \\ &+o\Big(\frac{n^{-\varepsilon}}{d}\Big) + O\Big((k+\frac{1}{t})\frac{|b|^3}{t^3}\Big)\Big\}. \end{aligned}$$

Now we compute the different summations over m. By (4·1), $\exp(-kmt)$ is close to $\exp(-k_0mt)$ if m is not too large, but we have again some computations to do to control this approximation. For $s = 0, 1, 2$:

$$\sum_{m=1}^{\infty} \frac{1}{m^s} \exp(-kmt) = \sum_{m=1}^{\infty} \frac{1}{m^s} \exp(-k_0mt) \exp(-(k-k_0)mt).$$

Since $e^x = \sum_{n=0}^{M} \frac{x^n}{n!} + \sum_{n=M+1}^{\infty} \frac{x^n}{n!}$, we have

$$\Big| e^x - \sum_{n=0}^{M} \frac{x^n}{n!} \Big| \leqslant |x|^{M+1} \sum_{n=M+1}^{\infty} \frac{|x|^{n-M-1}}{n!} \leqslant |x|^{M+1} e^{|x|}.$$

Thus we obtain

$$\sum_{m=1}^{\infty} \frac{1}{m^s} \exp(-kmt) = \sum_{m=1}^{\infty} \frac{1}{m^s} \exp(-k_0mt) \Big\{ 1 - (k-k_0)mt + \frac{1}{2}(k-k_0)^2 m^2 t^2$$
$$- \frac{1}{6}(k-k_0)^3 m^3 t^3 + O(|k-k_0|^4 m^4 t^4 \exp(|k-k_0|mt)).$$

By (2·2), $\exp(|k-k_0|mt) \leqslant \exp(mk_0t/2)$. Next we use the fact that $k_0t = \log 2$:

$$\begin{aligned}\sum_{m=1}^{\infty} \frac{1}{m^s} \exp(-kmt) &= \sum_{m=1}^{\infty} \frac{1}{m^s 2^m} \Big(1 - (k-k_0)mt + \frac{1}{2}(k-k_0)^2 m^2 t^2 \\ &\quad - \frac{1}{6}(k-k_0)^3 m^3 t^3 + O(|k-k_0|^4 m^4 t^4 2^{m/2})) \\ &= \sum_{m=1}^{\infty} \frac{1}{m^s 2^m} - (k-k_0)t \sum_{m=1}^{\infty} \frac{2^{-m}}{m^{s-1}} \\ &\quad + \frac{1}{2}(k-k_0)^2 t^2 \sum_{m=1}^{\infty} \frac{2^{-m}}{m^{s-2}} - \frac{(k-k)^3}{6} t^3 \sum_{m=1}^{\infty} \frac{2^{-m}}{m^{s-3}} \\ &\quad + O\Big(|k-k_0|^4 t^4 \sum_{m=1}^{\infty} \frac{2^{-m/2}}{m^{s-4}} \Big).\end{aligned}$$

We obtain for the function g_k

$$\begin{aligned}g_k(w) = f(w) \exp \Big\{ &- \frac{1}{t} \sum_{m=1}^{\infty} \frac{1}{m^2 2^m} + (k-k_0) \sum_{m=1}^{\infty} \frac{1}{m 2^m} - \frac{(k-k_0)^2 t}{2} \sum_{m=1}^{\infty} 2^{-m} \\ &+ O(|k-k_0|^3 t^2) + \frac{1}{2} \sum_{m=1}^{\infty} \frac{1}{m 2^m} + O(|k-k_0| t) \\ &+ ib \Big(\frac{1}{t^2} \sum_{m=1}^{\infty} \frac{1}{m^2 2^m} - \frac{(k-k_0)}{t} \sum_{m=1}^{\infty} \frac{1}{m 2^m} + O(|k-k_0|^2) \\ &+ \frac{k}{t} \sum_{m=1}^{\infty} \frac{1}{m 2^m} - k(k-k_0) \sum_{m=1}^{\infty} 2^{-m} + O(kt|k-k_0|^2) \Big) \\ &+ b^2 \Big(\frac{1}{t^3} \sum_{m=1}^{\infty} \frac{1}{m^2 2^m} + O\Big(\frac{|k-k_0|}{t^2}\Big) + \frac{k}{t^2} \sum_{m=1}^{\infty} \frac{1}{m 2^m} + O\Big(\frac{k|k-k_0|}{t}\Big) \\ &+ \frac{k^2}{2t} \sum_{m=1}^{\infty} 2^{-m} + O(k^2|k-k_0|) \Big) + o(n^{-\varepsilon} d^{-1}) + O\Big((k + \frac{1}{t}) \frac{|b|^3}{t^3} \Big) \Big\}.\end{aligned}$$

Next we compute the different sums on m:

$$\begin{aligned}
g_k(w) = f(w)\exp\Big\{ &-\frac{C_2}{t} + (k-k_0)\log 2 - \frac{(k-k_0)^2 t}{2} + \frac{1}{2}\log 2 \\
&+ ib\Big(\frac{C_2}{t^2} - \frac{(k-k_0)}{t}\log 2 + \frac{k\log 2}{t} - k(k-k_0)\Big) + b^2\Big(\frac{C_2}{t^3} + \frac{k}{t^2}\log 2 + \frac{k^2}{2t}\Big) \\
&+ O(|k-k_0|^3 t^2 + |k-k_0| t + |k-k_0|^2|b| + \frac{|k-k_0|b^2}{t^2}) \\
&+ o\Big(\frac{n^{-\varepsilon}}{d}\Big) + O\Big((k+\frac{1}{t})\frac{|b|^3}{t^3}\Big)\Big\}.
\end{aligned}$$

Then by (4·1) the above error terms give $o(n^{-\varepsilon}|\mathcal{D}|^{-1})$ and we can replace $-k(k-k_0)$ with $-k_0(k-k_0)$ in the coefficient of ib and analogously in that of b^2. Finally,

$$\begin{aligned}
g_k(w) = f(w)\exp\Big\{ &-\frac{C_2}{t} + (k-k_0)\log 2 - \frac{(k-k_0)^2 t}{2} + \frac{\log 2}{2} \\
&+ ib\Big(\frac{C_2}{t^2} + \frac{k_0\log 2}{t} - k_0(k-k_0)\Big) + b^2\Big(\frac{C_2}{t^3} + \frac{k_0\log 2}{t^2} + \frac{k_0^2}{2t}\Big) + o\Big(\frac{n^{-\varepsilon}}{|\mathcal{D}|}\Big)\Big\},
\end{aligned}$$

as claimed in Lemma 4.1. □

5. The term S_0, end of the proof of Theorem 1.1 in the case $\mathcal{D} = \{1, \ldots, d\}$.

As a special case of Lemma 4.1 applied with $\mathcal{D} = \{1, \ldots, d\}$, we remark that

$$g_k(dx_0) = f(dx_0)\exp\Big\{ -\frac{C_2}{dx_0} + (k-k_0)\log 2 - \frac{(k-k_0)^2 dx_0}{2} + \frac{\log 2}{2} + o\Big(\frac{n^{-\varepsilon}}{d}\Big)\Big\},$$

and

$$\begin{aligned}
\prod_{r=1}^{d} g_{N_r}(dx_0) = f^d(dx_0)\exp\Big\{ &-\frac{C_2}{x_0} + \sum_{r=1}^{d}(N_r - k_0)\log 2 \\
&- \frac{dx_0}{2}\sum_{r=1}^{d}(N_r-k_0)^2 + \frac{d\log 2}{2} + o(n^{-\varepsilon})\}.
\end{aligned} \tag{5·1}$$

Since $f(w) = \exp\left(\frac{\pi^2}{6w} + \frac{1}{2}\log\frac{w}{2\pi} + O(|w|)\right)$ for $w \to 0$ in $|\arg w| \leqslant \kappa < \pi/2$ and $\Re e\, w > 0$, we have for $|y| \leqslant y_1 \leqslant n^{-\frac{3}{4}+\frac{\varepsilon}{3}}$:

$$f(d(x_0+iy)) = \exp\Big(\frac{\pi^2}{6d(x_0+iy)} + \frac{1}{2}\log\Big(\frac{d(x_0+iy)}{2\pi}\Big) + O(dx_0)\Big),$$

$$\begin{aligned}
f^d(d(x_0+iy)) &= \exp\Big(\frac{\pi^2}{6(x_0+iy)} + \frac{d}{2}\log\Big(\frac{d(x_0+iy)}{2\pi}\Big) + O(d^2x_0)\Big) \\
&= \exp\Big(\frac{\pi^2}{6x_0}\Big(1 - \frac{iy}{x_0} - \frac{y^2}{x_0^2} + O(\frac{|y|^3}{x_0^3})\Big) + \frac{d}{2}\log\big(\frac{dx_0}{2\pi}\big) \\
&+ \frac{d}{2}\log\Big(1 + \frac{iy}{x_0}\Big) + O(d^2x_0)\Big) \\
&= \exp\Big(\frac{\pi^2}{6x_0} - iy\frac{\pi^2}{6x_0^2} - \frac{\pi^2 y^2}{6x_0^3} + \frac{d}{2}\log\big(\frac{dx_0}{2\pi}\big) + o(n^{-\varepsilon})\Big).
\end{aligned}$$

We obtain for the integrand

$$
\begin{aligned}
P &:= \Big\{ \prod_{r=1}^{d} g_{N_r}(d(x_0+iy)) \Big\} \exp((n-R-Q)(x_0+iy)) \\
&= \exp\Big\{ \frac{\pi^2}{6x_0} - iy\frac{\pi^2}{6x_0^2} - \frac{\pi^2 y^2}{6x_0^3} + \frac{d}{2}\log\big(\frac{dx_0}{2\pi}\big) + o(n^{-\varepsilon}) \\
&+ (n-R-Q)(x_0+iy) - \frac{C_2}{x_0} + \sum_{r=1}^{d}(N_r-k_0)\log 2 - \frac{1}{2}\sum_{r=1}^{d}(N_r-k_0)^2 dx_0 + \frac{d}{2}\log 2 \\
&+ idy\Big(\frac{C_2}{dx_0^2} + \frac{k_0\log 2}{x_0} - k_0\sum_{r=1}^{d}(N_r-k_0)\Big) + d^2y^2\Big(\frac{C_2}{d^2x_0^3} + \frac{k_0\log 2}{dx_0^2} + \frac{k_0^2}{2x_0}\Big) + o(n^{-\varepsilon}\Big\}.
\end{aligned}
$$

We collect terms in iy, y^2:

$$
\begin{aligned}
P = \exp\Big\{ & \frac{\pi^2}{6x_0} + \frac{d}{2}\log\big(\frac{dx_0}{2\pi}\big) + (n-R-Q)x_0 \\
& - \frac{C_2}{x_0} + \sum_{r=1}^{d}(N_r-k_0)\log 2 - \frac{1}{2}\sum_{r=1}^{d}(N_r-k_0)^2 dx_0 + \frac{d\log 2}{2} \\
& + iy\Big(n-R-Q - \frac{\pi^2}{6x_0^2} + \frac{C_2}{x_0^2} + \frac{dk_0\log 2}{x_0} - dk_0\sum_{r=1}^{d}(N_r-k_0)\Big) \\
& + y^2\Big(-\frac{\pi^2}{6x_0^3} + \frac{C_2}{x_0^3} + \frac{dk_0\log 2}{x_0^2} + \frac{d^2k_0^2}{2x_0}\Big) + o(n^{-\varepsilon})\Big\}.
\end{aligned}
$$

As a special case, we obtained that

$$
\begin{aligned}
\Big\{ \prod_{r=1}^{d} g_{N_r}(dx_0) \Big\} \exp((n-R-Q)x_0) = \exp\Big\{ & \frac{\pi^2}{6x_0} + \frac{d}{2}\log\big(\frac{dx_0}{2\pi}\big) + (n-R-Q)x_0 \\
& - \frac{C_2}{x_0} + \sum_{r=1}^{d}(N_r-k_0)\log 2 \\
& - \frac{1}{2}\sum_{r=1}^{d}(N_r-k_0)^2 dx_0 + \frac{d\log 2}{2} + o(n^{-\varepsilon})\Big\}.
\end{aligned}
\tag{5·2}
$$

Consequently,

$$
\begin{aligned}
& \Big\{ \prod_{r=1}^{d} g_{N_r}(d(x_0+iy)) \Big\} \exp((n-R-Q)(x_0+iy)) \\
&= \Big\{ \prod_{r=1}^{d} g_{N_r}(dx_0) \Big\} \exp((n-R-Q)x_0) \\
&\times \exp\Big\{ iy\Big(n-R-Q - \frac{\pi^2}{6x_0^2} + \frac{1}{x_0^2}\sum_{m=1}^{\infty}\frac{1}{m^2 2^m} + \frac{dk_0\log 2}{x_0} - dk_0\sum_{r=1}^{d}(N_r-k_0)\Big) \\
&+ y^2\Big(-\frac{\pi^2}{6x_0^3} + \frac{1}{x_0^3}\sum_{m=1}^{\infty}\frac{1}{m^2 2^m} + \frac{dk_0\log 2}{x_0^2} + \frac{d^2k_0^2}{2x_0}\Big) + o(n^{-\varepsilon})\Big\}.
\end{aligned}
\tag{5·3}
$$

The coefficient of y^2 in (5·3) is

$$\frac{1}{x_0^3}\Big(-\frac{\pi^2}{6}+\frac{\pi^2}{12}-\frac{\log^2 2}{2}+dk_0x_0\log 2+\frac{d^2k_0^2x_0^2}{2}\Big)=-\frac{2\sqrt{3}}{\pi}n^{\frac32}\Big(1-\frac{12\log^2 2}{\pi^2}\Big),$$

where $\frac{12\log^2 2}{\pi^2}<\frac{12\cdot 0.49}{\pi^2}<\frac{6}{\pi^2}<1$.

The coefficient of iy in (5·3) is

$$\begin{aligned}
&n-R-Q-\frac{1}{x_0^2}\Big(\frac{\pi^2}{6}-\sum_{m=1}^{\infty}\frac{1}{m^2 2^m}-dk_0x_0\log 2\Big)-dk_0\sum_{r=1}^{d}(N_r-k_0)\\
&=-R-Q+\frac{\log^2 2}{2x_0^2}-dk_0\sum_{r=1}^{d}(N_r-k_0)=-2dk_0\sum_{r=1}^{d}(N_r-k_0)+O(n^{\frac34-2\varepsilon}).
\end{aligned}$$

Since $|y|O(n^{\frac34-2\varepsilon})=o(n^{-\varepsilon})$ we infer from (5·3) that

$$\begin{aligned}
&\Big\{\prod_{r=1}^{d}g_{N_r}(d(x_0+iy))\Big\}\exp((n-R-Q)(x_0+iy))\\
&=\Big\{\prod_{r=1}^{d}g_{N_r}(dx_0)\Big\}\exp((n-R-Q)x_0)\\
&\times\exp\Big\{-iy2dk_0\sum_{r=1}^{d}(N_r-k_0)-y^2\frac{2\sqrt{3}n^{\frac32}}{\pi}\Big(1-\frac{12\log^2 2}{\pi^2}\Big)+o(n^{-\varepsilon})\Big\}.
\end{aligned}\tag{5·4}$$

Let $A=\frac{2\sqrt{3}n^{\frac32}}{\pi}\Big(1-\frac{12\log^2 2}{\pi^2}\Big)$, $(A>0)$ and $B=2dk_0\sum_{r=1}^{d}(N_r-k_0)$. Then, from (5·4)

$$\begin{aligned}
S_0&=\frac{d}{2\pi}\int_{-y_1}^{y_1}\Big\{\prod_{r=1}^{d}g_{N_r}(d(x_0+iy))\Big\}\exp((n-R-Q)(x_0+iy))\,\mathrm{d}y\\
&=\frac{d}{2\pi}\Big\{\prod_{r=1}^{d}g_{N_r}(dx_0)\Big\}\exp((n-R-Q)x_0)\int_{-y_1}^{y_1}\exp(-iyB-Ay^2+o(n^{-\varepsilon}))\,\mathrm{d}y.
\end{aligned}$$

Lemma 5.1. *We have :*

$$\int_{-y_1}^{y_1}\exp(-iyB-Ay^2+o(n^{-\varepsilon}))\,\mathrm{d}y=\sqrt{\frac{\pi}{A}}\exp\Big(-\frac{B^2}{4A}\Big)\Big\{1+o(n^{-\varepsilon})\exp\Big(\frac{B^2}{4A}\Big)\Big\}.\tag{5·5}$$

Proof. These are standard manipulations on Gaussian integrals thus we won't write all the details. Let $I_{AB}(y_1)$ be the integral of the left hand side of (5·5). Since for $|y|\leqslant y_1$, $\exp(-iyB-Ay^2+o(n^{-\varepsilon}))=(1+o(n^{-\varepsilon}))\exp(-iyB-Ay^2))$, we have :

$$\begin{aligned}
I_{AB}(y_1)&=\int_{-\infty}^{+\infty}\exp(-iyB-Ay^2)\,\mathrm{d}y\\
&\quad+O\Big(\int_{y_1}^{+\infty}\exp(-Ay^2)\,\mathrm{d}y\Big)+o(n^{-\varepsilon})\int_{-\infty}^{+\infty}\exp(-Ay^2)\,\mathrm{d}y.
\end{aligned}$$

The main term is a Gaussian integral :

$$\int_{-\infty}^{+\infty}\exp(-iyB-Ay^2)\,\mathrm{d}y=\sqrt{\frac{\pi}{A}}\exp\Big(-\frac{B^2}{4A}\Big).$$

For the error terms we have

$$\int_{y_1}^{+\infty} \exp(-Ay^2)\,\mathrm{d}y \ll \frac{1}{Ay_1}\exp(-Ay_1^2) \text{ and } \int_{-\infty}^{+\infty} \exp(-Ay^2) \ll \frac{1}{\sqrt{A}},$$

the Lemma follows.

Furthermore

$$\frac{B^2}{4A} = O\Big(n^{-\frac{3}{2}}\Big(\sqrt{n}d\frac{n^{\frac14}\sqrt{\log n}}{dw(n)}\Big)^2\Big) = o(\log n).$$

Thus the error term in Lemma 5.1 is :

$$o(n^{-\varepsilon})\exp\Big(\frac{B^2}{4A}\Big) = o(1).$$

Therefore

$$S_0 = (1+o(1))\sqrt{\frac{\pi}{A}}\exp\Big(-\frac{B^2}{4A}\Big)\frac{d}{2\pi}\Big\{\prod_{r=1}^{d} g_{N_r}(dx_0)\Big\}\exp((n-R-Q)x_0).$$

Adding the estimates for the trivial parts (see (2·8), (3·1)) we obtain that for $n \equiv R \,(\mathrm{mod}\, d)$,

$$\begin{aligned}\Pi_d^*(n,\mathcal{R}) &= \Big\{\prod_{r=1}^{d} g_{N_r}(dx_0)\Big\}\exp((n-R-Q)x_0)\\ &\quad\times\Big\{(1+o(1))\sqrt{\frac{\pi}{A}}\exp\Big(-\frac{B^2}{4A}\Big)\frac{d}{2\pi} + O\Big(\exp\Big(-\frac{1+o(1)}{4x_0}\Big)\Big)\\ &\quad + O\Big(\exp\Big(-\frac{\sqrt{3}}{\pi^5 3^3}n^{\frac14+2\varepsilon}\Big)\Big) + O\Big(\exp\Big(-\frac{\sqrt{3}}{\pi^5 3^3}n^{2\varepsilon/3}\Big)\Big)\Big\}\\ &= (1+o(1))\sqrt{\frac{\pi}{A}}\exp\Big(-\frac{B^2}{4A}\Big)\frac{d}{2\pi}\Big\{\prod_{r=1}^{d} g_{N_r}(dx_0)\Big\}\exp((n-R-Q)x_0).\end{aligned} \tag{5·6}$$

To end the proof, it remains to insert the classical formula

$$q(n) = (1+o(1))\frac{1}{4.3^{1/4}n^{3/4}}\exp\Big(\frac{\pi\sqrt{n}}{\sqrt{3}}\Big), \tag{5·7}$$

and our previous results on the $g_{N_r}(dx_0)$ (see (5·2)), and to do the convenient computations. We obtain

$$\begin{aligned}\Pi_d^*(n,\mathcal{R}) &= (1+o(1))4.3^{1/4}q(n)\frac{1}{\sqrt{1-\frac{12(\log 2)^2}{\pi^2}}}\frac{d}{2\sqrt{2}3^{1/4}}\Big(\frac{d}{2\sqrt{3n}}\Big)^{d/2}\exp\Big(-\frac{B^2}{4A}\\ &\quad+\sum_{r=1}^{d}(N_r-k_0)\log 2 - x_0\big(R+Q-\frac{\log^2 2}{2x_0^2}\big) - \frac12\sum_{r=1}^{d}(N_r-k_0)^2 dx_0\Big).\end{aligned} \tag{5·8}$$

By formulae (2.11) and (2.12) of [6] and by (2·4), we have:

$$R+Q-\frac{\log^2 2}{2x_0^2} = \frac{dk_0}{2} + dk_0\sum_{r=1}^{d}(N_r-k_0) + \frac{d}{2}\sum_{r=1}^{d}(N_r-k_0)^2 + o(\sqrt{n}).$$

Thus the argument of the exponential in (5·8) is

$$\exp\Big(-\frac{B^2}{4A}+...\Big) = \exp\Big(-\frac{B^2}{4A} - \frac{\log 2}{2} - dx_0\sum_{r=1}^{d}(N_r-k_0)^2 + o(1)\Big).$$

Inserting this in (5·8) ends the proof of Theorem 1.1 for $\mathcal{D} = \{1,\ldots,d\}$.

6. First steps of the proof of Theorem 1.1 for $\mathcal{D} \neq \{1, \dots, d\}$

Like in Section 3 of [6], we apply Lemma 2.1 of [6], the Cauchy formula and write $z = x_0 + iy$:

$$\begin{aligned}\Pi_d^*(n, \mathcal{R}_\mathcal{D}) = &\frac{1}{2\pi}\int_{-\pi}^{\pi} \Big\{ \prod_{r \in \mathcal{D}^c} h_r(x_0 + iy)\Big\} \\ &\times \Big\{ \prod_{r\in\mathcal{D}} g_{N_r}(d(x_0+iy))\Big\} \exp\big((n - \mathcal{R}_\mathcal{D} - Q_\mathcal{D})(x_0+iy)\big)\,\mathrm{d}y,\end{aligned} \tag{6·1}$$

with

$$h_r(z) = \prod_{j=0}^{\infty}(1 + \exp(-(r+jd)z)).$$

When $\mathcal{D}^c \neq \varnothing$ and $\mathcal{D}^c \neq \{d\}$, the functions h_r are not $2\pi/d$-periodic but we still split the integral in intervales of length $2\pi/d$ in order to use our previous work on the functions g_k.

$$\Pi_d^*(n, \mathcal{R}_\mathcal{D}) = \frac{1}{2\pi}\sum_{|\lambda| \leqslant \lfloor \frac{d-1}{2}\rfloor} \int_{-\frac{\pi}{d}+\frac{2\lambda\pi}{d}}^{\frac{\pi}{d}+\frac{2\lambda\pi}{d}} \dots + B,$$

with

$$B = \begin{cases} 0 & \text{if } d \text{ is odd} \\ \int_{-\pi}^{-\pi+\frac{\pi}{d}} \dots + \int_{\pi-\frac{\pi}{d}}^{\pi} \dots = \int_{\pi-\frac{\pi}{d}}^{\pi+\frac{\pi}{d}} & \text{if } d \text{ is even.}\end{cases}$$

Next we do some convenient change of variables :

$$\begin{aligned}\Pi_d^*(n, \mathcal{R}_\mathcal{D}) = &\frac{1}{2\pi}\sum_{-\frac{d}{2}<\lambda\leqslant\frac{d}{2}} \int_{-\frac{\pi}{d}}^{\frac{\pi}{d}} \Big\{ \prod_{r\in\mathcal{D}^c} h_r(x_0 + iy + i\frac{2\lambda\pi}{d})\Big\}\Big\{\prod_{r\in\mathcal{D}} g_{N_r}(d(x_0+iy))\Big\} \\ &\times \exp((n - \mathcal{R}_\mathcal{D} - Q_\mathcal{D})(x_0+iy) + (n-\mathcal{R}_\mathcal{D})\frac{2i\lambda\pi}{d})\,\mathrm{d}y \\ = &\sum_{-\frac{d}{2}<\lambda\leqslant\frac{d}{2}} S(\lambda),\end{aligned}$$

say. We will write the splitting

$$S(\lambda) = S_0(\lambda) + S_1(\lambda) + S_2(\lambda),$$

where in $S_0(\lambda)$ the range of integration for y is $|y| \leqslant y_1$, in $S_1(\lambda)$ it is for $y_1 \leqslant |y| \leqslant y_2$ (with $y_2 = 3\pi x_0$) and in $S_2(\lambda)$ we take $y_2 \leqslant |y| \leqslant \frac{\pi}{d}$ (cf. (4.11) of [6]).

7. Upper bounds of $S_1(\lambda)$ and $S_2(\lambda)$

To obtain a convenient upper bound of these terms we first remark that

$$|h_r(x_0 + iy + \frac{2i\pi\lambda}{d})| \leqslant h_r(x_0). \tag{7·1}$$

Let $j \in \mathbb{N}$. To prove (7·1) it is enough to prove that each $T_j \leqslant 1$ with

$$T_j := \frac{\Big|1 + \exp\Big(-(r+jd)(x_0 + iy + \frac{2i\pi\lambda}{d})\Big)\Big|}{|1 + \exp(-(r+jd)x_0)|}.$$

By a simple computation we have

$$T_j^2 = 1 - \frac{4\exp(-x_0(r+jd))\sin^2\big(\frac{y}{2}(r+jd)+\frac{\pi\lambda r}{d}\big)}{(1+\exp(-x_0(r+jd)))^2} \leqslant 1.$$

By Lemma 2.1 we have

$$|S_2(\lambda)| \leqslant \Big\{\prod_{r\in\mathcal{D}} g_{N_r}(dx_0)\Big\}\Big\{\prod_{r\in\mathcal{D}^c} h_r(x_0)\Big\}\exp((n-R_{\mathcal{D}}-Q_{\mathcal{D}})x_0)\exp\Big(\frac{|\mathcal{D}|(-1+o(1))}{4dx_0}\Big).$$

By Lemma 3.1 we also have

$$|S_1(\lambda)| \leqslant \Big\{\prod_{r\in\mathcal{D}} g_{N_r}(dx_0)\Big\}\Big\{\prod_{r\in\mathcal{D}^c} h_r(x_0)\Big\}\exp((n-R_{\mathcal{D}}-Q_{\mathcal{D}})x_0)\exp\Big(-\frac{|\mathcal{D}|\sqrt{3}n^{2\varepsilon/3}}{27\pi^5 d}\Big).$$

If $|\mathcal{D}|$ is small, *i.e.*, if $|\mathcal{D}| \leqslant dn^{-\varepsilon/3}$ this last estimate for $S_1(\lambda)$ is not sufficient. However, by Lemma 3.1 (i), the contribution of the range $n^{-5/8+\varepsilon} \leqslant |y| \leqslant 3\pi x_0$ to $S_1(\lambda)$ is

$$\leqslant \Big\{\prod_{r\in\mathcal{D}} g_{N_r}(dx_0)\Big\}\Big\{\prod_{r\in\mathcal{D}^c} h_r(x_0)\Big\}\exp((n-R_{\mathcal{D}}-Q_{\mathcal{D}})x_0)\exp\Big(-\frac{|\mathcal{D}|\sqrt{3}n^{1/4+2\varepsilon}}{27\pi^5 d}\Big).$$

Thus it remains to handle the case $|\mathcal{D}| \leqslant dn^{-\varepsilon/3}$ in the range $y_1 \leqslant |y| < n^{-5/8+\varepsilon}$.

• First we study the case $\lambda = 0$. We use a similar argument as in the proof of Lemma 3.1. For any $1 \leqslant r \leqslant d$, let J_r denote the set of the integers j such that

$$\frac{1}{2x_0} \leqslant r + jd \leqslant \frac{1}{x_0}. \tag{7·2}$$

Then for $j \in J_r$ we have

$$\frac{\exp(-x_0(r+jd))}{(1+\exp(-x_0(r+jd)))^2} \geqslant \frac{1}{\mathrm{e}(1+\mathrm{e}^{-1/2})^2}. \tag{7·3}$$

This gives for the correspondent T_j :

$$T_j^2 \leqslant 1 - \frac{4}{\mathrm{e}(1+\mathrm{e}^{-1/2})^2}\sin^2\big(\frac{y}{2}(r+jd)\big).$$

Next, quite like in the proof of Lemma 3.1, we have for n large enough

$$\begin{aligned}|h_r(x_0+iy)| &\leqslant h_r(x_0)\Big(1 - \frac{4}{\mathrm{e}(1+\mathrm{e}^{-1/2})^2}\frac{y_1^2}{4\pi^2 x_0^2}\Big)^{|J_r|/2} \\ &\leqslant h_r(x_0)\exp\Big(-\frac{y_1^2}{48\pi^2 dx_0^3}\Big) \leqslant h_r(x_0)\exp\Big(-\frac{\sqrt{3}n^{2\varepsilon/3}}{2\pi^5 d}\Big).\end{aligned}$$

This upper bound combined with Lemma 3.1 is sufficient to obtain

$$|S_1(0)| \leqslant \Big\{\prod_{r\in\mathcal{D}} g_{N_r}(dx_0)\Big\}\Big\{\prod_{r\in\mathcal{D}^c} h_r(x_0)\Big\}\exp((n-R_{\mathcal{D}}-Q_{\mathcal{D}})x_0)\exp\Big(-\frac{\sqrt{3}n^{2\varepsilon/3}}{27\pi^5}\Big).$$

• Now we suppose that $\lambda \neq 0$. We write $\frac{\lambda}{d} = \frac{\lambda'}{d'}$ with $(\lambda', d') = 1$ and $d' > 0$.

First we suppose that $d' \geqslant n^{\varepsilon/4}$. Since $(\lambda', d') = 1$, there are $\frac{d'}{4} + O(1)$ integers $r_0 \in \{1, \dots, d'\}$ such that $\frac{\lambda' r_0}{d'} \bmod 1 \in [\frac{1}{4}, \frac{1}{2}]$. Thus there are $\frac{d}{4} + O(\frac{d}{d'})$ integers $r \in \{1, \dots, d\}$ such that $\frac{\lambda' r}{d'} \bmod 1 \in [\frac{1}{4}, \frac{1}{2}]$ (again for n large enough).

Since $|\mathcal{D}| \leqslant dn^{-\varepsilon/3} < \frac{d}{4} + O\left(\frac{d}{d'}\right)$ for n large enough, there exists $r_1 \in \mathcal{D}^c$ such that $\frac{\lambda' r_1}{d'} \bmod 1 \in [\frac{1}{4}, \frac{1}{2}]$. For $j \in J_{r_1}$, $|(r_1 + jd)y| \ll n^{-1/8+\varepsilon}$. Thus $\sin^2((r_1 + jd)\frac{y}{2} + \frac{\lambda r_1 \pi}{d})) \geqslant \sin^2 \frac{\pi}{6} = \frac{1}{4}$.

This gives

$$\prod_{j \in J_{r_1}} |T_j(r_1)|^2 \leqslant \left(\frac{11}{12}\right)^{|J_{r_1}|} \leqslant \exp(-\frac{1}{30dx_0}),$$

which is a sufficient upper bound.

Now we suppose that $2 \leqslant d' < n^{\varepsilon/4}$. There are $d - \frac{d}{d'}$ integers r such that $d' \nmid r$. For these integers r and $j \in J_r$, we have

$$\sin^2((r + jd)\frac{y}{2} + \frac{\pi \lambda r}{d}) \geqslant \sin^2\left(\frac{\pi}{2d'}\right) \geqslant \frac{1}{d^2}.$$

Since $|\mathcal{D}| \leqslant dn^{-\varepsilon/3}$, there are at least $d/3$ such integers $r \in \mathcal{D}^c$ such that $d' \nmid r$.

Thus we have :

$$\prod_{r \in \mathcal{D}^c} \frac{|h_r(x_0 + iy + \frac{2i\pi\lambda}{d})|}{h_r(x_0)} \leqslant \prod_{\substack{r \in \mathcal{D}^c \\ r \not\equiv 0 \,(\bmod d')}} \prod_{j \in J_r} \left(1 - \frac{1}{6d^2}\right) \leqslant \exp\left(-\frac{d}{48d^3 x_0}\right),$$

which is a sufficient upper bound when $d \leqslant n^{1/4-2\varepsilon}$.

8. The terms $S_0(\lambda)$ for $\lambda \neq 0$

We have to consider the integrals

$$S_0(\lambda) = \frac{1}{2\pi} \int_{|y| \leqslant y_1} \Big\{ \prod_{r \in \mathcal{D}^c} h_r(x_0 + iy + \frac{2i\pi\lambda}{d}) \Big\} \Big\{ \prod_{r \in \mathcal{D}} g_{N_r}(d(x_0 + iy)) \Big\}$$
$$\times \exp((n - R_{\mathcal{D}} - Q_{\mathcal{D}})(x_0 + iy + \frac{2i\pi\lambda}{d}))\, \mathrm{d}y.$$

• First we suppose that there exists $r \in \mathcal{D}^c$ such that $\lambda r \not\equiv 0 \,(\bmod d)$. In the previous section we remarked that

$$|h_r(x_0 + iy + \frac{2i\pi\lambda}{d})| = h_r(x_0) \prod_{j=0}^{\infty} \Big(1 - \frac{4\exp(-x_0(r + jd))\sin^2\left(\frac{y}{2}(r + jd) + \frac{\pi\lambda r}{d}\right)}{(1 + \exp(-x_0(r + jd))^2}\Big)^{1/2}.$$

We work again with the sets J_r defined by (7·2) in the previous section. For $j \in J_r$, $\lambda r \equiv a \,(\bmod d), 1 \leqslant |a| \leqslant d/2$ and $|y| \leqslant y_1$ we have:

$$\sin^2\Big(\frac{y}{2}(r+jd)+\frac{\pi\lambda r}{d}\Big) = \sin^2\Big(\frac{\pi|a|}{d} \pm \frac{|y|}{2}(r+jd)\Big) \geqslant \sin^2\Big(\frac{\pi|a|}{d} - \frac{y_1}{x_0}\Big) \geqslant \sin^2\Big(\frac{\pi}{d} - \frac{y_1}{x_0}\Big) \geqslant \frac{3}{d^2},$$

for n large enough (we recall that $d \leqslant n^{1/4-2\varepsilon}$).

Since $|J_r| = \frac{1}{2dx_0} + O(1) \geqslant \frac{1}{3dx_0}$, we have:

$$\begin{aligned}
|h_r(x_0 + iy + \frac{2i\pi\lambda}{d})| &\leqslant h_r(x_0) \prod_{j \in J_r} \Big(1 - \frac{12}{\mathrm{e}d^2(1 + \mathrm{e}^{-1/2})^2}\Big)^{1/2} \\
&\leqslant h_r(x_0) \exp\Big(\frac{1}{6dx_0} \log\big(1 - \frac{12}{\mathrm{e}d^2(1 + \mathrm{e}^{-1/2})^2}\big)\Big) \\
&\leqslant h_r(x_0) \exp\Big(-\frac{2}{\mathrm{e}d^3 x_0(1 + \mathrm{e}^{-1/2})^2}\Big).
\end{aligned}$$

Thus if there exists $r_0 \in \mathcal{D}^c$ such that $\lambda r_0 \not\equiv 0 \,(\mathrm{mod}\, d)$ then
(8·1)
$$|S_0(\lambda)| \leqslant \prod_{r\in\mathcal{D}} g_{N_r}(dx_0) \prod_{r\in\mathcal{D}^c} h_r(x_0) \exp((n-R_{\mathcal{D}}-Q_{\mathcal{D}})x_0) \exp\Big(-\frac{2}{\mathrm{e}d^3x_0(1+\mathrm{e}^{-1/2})^2}\Big).$$

This upper bound is sufficient only for $d \leqslant n^{1/6-\varepsilon}$.

• We suppose now that

$$\lambda r \equiv 0 \,(\mathrm{mod}\, d) \text{ for all } r \in D^c. \tag{8·2}$$

If $\mathcal{D}^c = \{d\}$ then by (1·1), $n - R_D - Q_{\mathcal{D}} \equiv 0 \,(\mathrm{mod}\, d)$ and $S_0(\lambda) = S_0(0)$ for all λ.

In the general case, if (8·2) holds then we must have $\frac{d}{(\lambda,d)}|r$. Thus $\frac{d}{(\lambda,d)}|\delta$ and again by (1·1), we have $n - R_{\mathcal{D}} - Q_{\mathcal{D}} \equiv 0 \,(\mathrm{mod}\, \frac{d}{(\lambda,d)})$. Thus $\exp((n-R_{\mathcal{D}}-Q_{\mathcal{D}})\frac{2i\pi\lambda}{d}) = 1$ and $S_0(\lambda) = S_0(0)$.

For $\mathcal{D}^c$ given, there exists δ integers λ modulo d such that $r\lambda \equiv 0 \,(\mathrm{mod}\, d)$ for all $r \in \mathcal{D}^c$.

We summarize these observations in the following lemma.

Lemma 8.1. *For $d \leqslant n^{1/6-\varepsilon}$ we have:*

$$\sum_{-\frac{d}{2}<\lambda\leqslant\frac{d}{2}} S_0(\lambda) = \delta S_0(0)(1+o(1)).$$

9. The function h_r^{-1} in the range $|y| \leqslant y_1$

The generating function associated to unequal partitions is $h(z) = \prod_{j=1}^{\infty}(1+z^j)$. For $S_0(0)$ it remains to handle the integral

$$\frac{1}{2\pi}\int_{|y|\leqslant y_1} h(\exp(-(x_0+iy)))\Big\{\prod_{r\in\mathcal{D}} \frac{g_{N_r}(d(x_0+iy))}{h_r(x_0+iy)}\Big\} \exp((n-R_{\mathcal{D}}-Q_{\mathcal{D}})(x_0+iy))\,\mathrm{d}y.$$

In this section we will state an asymptotic estimate related to h_r^{-1} in the range $|y| \leqslant y_1$.

The following method looks like the general method of Meinardus [7] for studying generating functions associated to partitions (this method is presented in details in the book of Andrews [1]). In fact we were also inspired by the chapter on the application of saddle point method to the partitions function in the Master course of Tenenbaum [8].

In Lemma 4.1, we obtained an estimation of $g_k(dw)$ in function of $f(dw)$. This leads us to try in fact to obtain an estimation of $U_r(z) := f(dz)h_r^{-1}(z)$ for $|y| \leqslant y_1$ instead of $h_r^{-1}(z)$. This change will make some computations easier. Thus we consider

$$U_r(z) = \prod_{j=1}^{\infty}(1-\exp(-jdz))^{-1}\prod_{j=0}^{\infty}(1+\exp(-(r+jd)z))^{-1}.$$

The main result of this section is the following lemma

Lemma 9.1. *Let $\eta > 0$. For $|y| \leqslant y_1$ and $1 \leqslant r \leqslant d$, we have*

$$U_r(z) = \exp\Big(\frac{\pi^2}{12dz} + \big(\frac{r}{d}-1\big)\log 2 + \frac{1}{2}\log\Big(\frac{dz}{\pi}\Big) + O(d^{1+\eta}r^{-\eta}|z|) + O(d^{-1}n^{-2\varepsilon})\Big).$$

In the next section we will apply this lemma with $\eta > 0$ small enough such that $d^{2+\eta}|z| \leqslant n^{-\varepsilon}$.

Proof. If $r=d$, there are quite no work to do since

$$U_d(z)=\prod_{j=1}^{\infty}(1+\exp(-jdz))^{-1}(1-\exp(-jdz))^{-1}=\prod_{j=1}^{\infty}(1-\exp(-2jdz))^{-1}=f(2dz).$$

Thus

$$U_d(z)=\exp\Big(\frac{\pi^2}{12dz}+\frac{1}{2}\log\big(\frac{2dz}{2\pi}\big)+O(d|z|)\Big).$$

Now we suppose that $r\neq d$. We prefer to work with

$$\begin{aligned}\tilde{U}_r(z):=&\prod_{j=1}^{\infty}(1-\exp(-jdz))^{-1}(1+\exp(-(r+jd)z))^{-1}\\ &=U_r(z)(1+\exp(-rz))=U_r(z)u_r(z).\end{aligned}$$

We easily see that

$$u_r(z)=(2+O(r|z|)). \tag{9·1}$$

Let $F(v,s)=\sum_{k=1}^{\infty}\frac{\exp(-kv)}{k^s}$. If $\Re e\, v>0$, then $s\mapsto F(v,s)$ is analytic on $\mathbb{C}$.
The first step of the proof is the following result.

Lemma 9.2. *For $r\neq d$, $\eta>0$, $z=x_0+iy$ with $|y|\leqslant y_1$, we have*

$$\begin{aligned}\log(\tilde{U}_r(z))=&\frac{1}{dz}\Big(\frac{\pi^2}{6}+F(rz+i\pi,2)\Big)+\frac{1}{2}\log\big(\frac{dz}{2\pi}\big)\\ &-\frac{1}{2}F(rz+i\pi,1)+\frac{dz}{12}\Big(-\frac{1}{2}+F(z+i\pi,0)\Big)+O(|z|r^{-\eta}d^{1+\eta}).\end{aligned}$$

The last term $\frac{dz}{12}\Big(-\frac{1}{2}+F(z+i\pi,0)\Big)$ in the above formula could be removed because it is $O(|z|r^{-\eta}d^{1+\eta})$. The following proof uses Mellin formula and looks like the general method of Meinardus for studying generating functions associated to partitions.

We begin by some standard manipulations

$$\begin{aligned}\tilde{U}_r(z)&=\exp\Big\{-\sum_{j=1}^{\infty}(\log(1+\exp(-(r+jd)z))+\log(1-\exp(-jdz))\Big\}\\ &=\exp\Big\{\sum_{j,k\geqslant1}\frac{\exp(-jkdz)}{k}((-1)^k\exp(-krz)+1)\Big\}.\end{aligned}$$

Let us write

$$\log(\tilde{U}_r(z))=\sum_{m=1}^{\infty}\frac{\beta(m)}{m}\exp(-mdz),$$

with

$$\beta(m)=\sum_{jk=m}j((-1)^k\exp(-krz)+1).$$

By the Mellin transform formula we have:

$$\log(\tilde{U}_r(z))=\sum_{m\geqslant1}\frac{\beta(m)}{m}\cdot\frac{1}{2i\pi}\int_{2-i\infty}^{2+i\infty}\frac{\Gamma(s)}{(mdz)^s}\,\mathrm{d}s.$$

But the Dirichlet series is

$$\begin{aligned}\sum_{m\geqslant 1}\frac{\beta(m)}{m^{s+1}}&=\sum_{j\geqslant 1}\frac{1}{j^s}\sum_{k\geqslant 1}\frac{(1+(-1)^k\exp(-krz))}{k^{s+1}}\\&=\zeta(s)(\zeta(s+1)+F(rz+i\pi,s+1)).\end{aligned}$$

Thus we have

$$\log(\tilde U_r(z))=\frac{1}{2i\pi}\int_{2-i\infty}^{2+i\infty}\frac{\Gamma(s)\zeta(s)(\zeta(s+1)+F(rz+i\pi,s+1))}{(dz)^s}\,\mathrm{d}s.$$

Let $\eta\in]0,1[$. We move the integral until the line $\Re e\, s=-1-\eta$. This gives

$$\begin{aligned}\log(\tilde U_r(z))=&\frac{1}{2i\pi}\int_{-1-\eta-i\infty}^{-1-\eta+i\infty}\frac{\Gamma(s)\zeta(s)(\zeta(s+1)+F(rz+i\pi,s+1))}{(dz)^s}\,\mathrm{d}s\\&+Res(1)+Res(0)+Res(-1)+E,\end{aligned}$$

where E is the error term arising from the horizontal branches.

For $\Re e\, s\in[-1-\eta,2]$ we have:

$$\begin{aligned}F(zr+i\pi,s+1)&=\sum_{\ell\geqslant 1}\mathrm{e}^{-2\ell zr}\Big(\frac{1}{(2\ell)^{s+1}}-\frac{\mathrm{e}^{rz}}{(2\ell-1)^{s+1}}\Big)\\&=\sum_{\ell\geqslant 1}\mathrm{e}^{-2\ell zr}\Big(\frac{1}{(2\ell)^{s+1}}-\frac{1}{(2\ell-1)^{s+1}}\Big)+O\Big(r|z|\sum_{\ell\geqslant 1}\frac{\mathrm{e}^{-2\ell rx_0}}{\ell^{1+\Re es}}\Big).\end{aligned}$$

When ℓ is small, the term $\mathrm{e}^{-\ell rx_0}$ is $O(1)$. Thus for $-1-\eta\leqslant\Re e\, s\leqslant 2$, we have :

$$\sum_{\ell\geqslant 1}\frac{\mathrm{e}^{-2\ell rx_0}}{\ell^{1+\Re e\, s}}\ll\sum_{\ell\leqslant 100/rx_0}\ell^{\eta}+\sum_{\ell>100/rx_0}\ell^{\eta}\mathrm{e}^{-2\ell rx_0}\ll(rx_0)^{-1-\eta},$$

since for the second sum we have

$$\sum_{\ell>100/rx_0}\ell^{\eta}\mathrm{e}^{-2\ell rx_0}\ll\sum_{\ell>100/rx_0}\ell\mathrm{e}^{-2\ell rx_0}(rx_0)^{1-\eta}\ll\frac{(rx_0)^{1-\eta}}{(1-\exp(-2rx_0))^2}.$$

For the other term we have

$$\Big|\sum_{\ell\geqslant 1}\mathrm{e}^{-2\ell rz}\Big(\frac{1}{(2\ell)^{1+s}}-\frac{1}{(2\ell-1)^{1+s}}\Big)\Big|\ll(1+|s|)\sum_{\ell\geqslant 1}\frac{\mathrm{e}^{-2\ell rx_0}}{\ell^{1-\eta}}\ll\frac{1+|s|}{(rx_0)^{\eta}}.$$

Thus for $-1-\eta\leqslant\Re e\, s\leqslant 2$, we have

$$|F(zr+i\pi,s+1)|\ll|s|(rx_0)^{-\eta}+r|z|(rx_0)^{-1-\eta}.\tag{9·2}$$

We will also use the following classical results for the functions ζ and Γ in vertical strips:

- there exists $H>0$ such that $|\zeta(s)|\ll|\Im m\, s|^H$ for $-3\leqslant\Re e\, s\leqslant 3$ (in fact more generally for $\Re e\, s\in[\sigma_1,\sigma_2]$) and $|\Im m\, s|\geqslant 1$ (cf. [2] Theorem 12.23 p. 270 for a more precise formulation) ;

- for $-3 \leqslant \Re e\, s \leqslant 3$ (or $\Re e\, s \in [\sigma_1, \sigma_2]$) and $|\Im m\, s| \to +\infty$, ([9] Corollaire II.0.13 p. 182) we have

$$\Gamma(s) = (1 + O(|\Im m\, s|^{-1}))\sqrt{2\pi}|\Im m\, s|^{\Re e\, s - \frac{1}{2}} \mathrm{e}^{-\pi|\Im m\, s|/2}\mathrm{e}^{i\alpha(s)},$$

with $\alpha(s) = (\Im m\, s)\log|\Im m\, s| - \Im m\, s + \frac{1}{2}\pi(\Re e\, s - \frac{1}{2})\operatorname{sgn}(\Im m\, s)$.

With this two formulae and by (9·2) we easily see that

$$\lim_{T\to+\infty}\int_{-1-\eta\pm iT}^{2\pm iT} \frac{\Gamma(s)\zeta(s)(\zeta(s+1) + F(rz + i\pi, s+1))}{(dz)^s}\,\mathrm{d}s = 0, \tag{9·3}$$

and

$$\Big|\int_{\Re e\, s=-1-\eta} \frac{\Gamma(s)\zeta(s)(\zeta(s+1) + F(rz + i\pi, s+1))}{(dz)^s}\,\mathrm{d}s\Big| \ll r^{-\eta}d^{1+\eta}|z|. \tag{9·4}$$

Now we compute the different residues:
We have

$$Res(1) = \frac{\Gamma(1)\big(\zeta(2) + F(rz + i\pi, 2)\big)}{dz} = \frac{1}{dz}\big(\frac{\pi^2}{6} + F(rz + i\pi, 2)\big). \tag{9·5}$$

In $s = 0$, we have two poles, one from Γ, the other from the function $s \mapsto \zeta(s+1)$. We use the well known results $\Gamma'(1) = -\gamma$, $\zeta(s) = \frac{1}{s-1} + \gamma + O(|s-1|)$).

Thus $Res(0)$ is the coefficient in s^{-1} in the following formula:

$$\begin{aligned}&\Big(\frac{1 - \gamma s + O(|s|^2)}{s}\Big)(\zeta(0) + s\zeta'(0) + O(|s|^2))\\ &\times \Big[\frac{1}{s} + \gamma + O(|s|) + F(rz + i\pi, 1)\Big](1 - s\log(dz) + O(|s|^2)).\end{aligned}$$

Since $\zeta(0) = -\frac{1}{2}$, $\zeta'(0) = -\frac{1}{2}\log(2\pi)$, we find

$$Res(0) = \frac{1}{2}\Big(\log\big(\frac{dz}{2\pi}\big) - F(rz + i\pi, 1)\Big). \tag{9·6}$$

In $s = -1$, Γ has a simple pole with residue -1 thus we have:

$$Res(-1) = -\zeta(-1)(\zeta(0) + F(rz + i\pi, 0))dz.$$

Since $\zeta(-1) = -\frac{1}{12}$, we obtain

$$Res(-1) = \frac{dz}{12}\big(-\frac{1}{2} + F(rz + i\pi, 0)\big). \tag{9·7}$$

Formulae (9·5), (9·6), (9·7), (9·3), (9·4) end the proof of Lemma 9.2.

Now we have to study the terms $F(*, s)$ with $s = 0, 1, 2$. First we have

$$F(rz + i\pi, 0) = \sum_{m=1}^{\infty}(-1)^m\exp(-mrz) = -\frac{1}{1 + \exp(rz)} = -\frac{1}{2} + O(rn^{-1/2}),$$

for $|y| \leqslant y_1$. For the contribution of $F(rz + i\pi, 1)$ we have

$$\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{2m}\exp(-mrz) = \frac{1}{2}\log\Big(1 + \exp(-rz)\Big) = \frac{1}{2}\log 2 + O(r|z|). \tag{9·8}$$

The contribution of $F(rz + i\pi, 2)$ is more difficult to handle because of the factor $(dz)^{-1}$. We will prove the following lemma

Lemma 9.3. *For $|y| \leqslant y_1$, we have for $d \leqslant n^{1/4-2\varepsilon}$*

$$\frac{1}{dz}F(rz+i\pi,2) = -\frac{\pi^2}{12dz} + \frac{r}{d}\log 2 + O(d^{-1}n^{-2\varepsilon}).$$

Proof. Let M be an odd integer. We have

$$\frac{1}{dz}F(rz+i\pi,2) = \frac{1}{dz}\sum_{m=1}^{M}\frac{(-1)^m\exp(-mrz)}{m^2} + \sum_{m=M+1}^{+\infty}\frac{(-1)^m\exp(-mrz)}{dzm^2} = T_1+T_2,$$

say. We begin with T_2. Since the sum is absolutely convergent, we can regroup the terms $m=2\ell$ with the terms $m=2\ell+1$

$$\begin{aligned}|T_2| &\leqslant \frac{1}{d|z|}\sum_{\ell=\frac{M+1}{2}}^{+\infty}\exp(-2\ell x_0)\Big|\frac{1}{4\ell^2}-\frac{\exp(-rz)}{(2\ell+1)^2}\Big| \leqslant \frac{1}{d|z|}\sum_{\ell=\frac{M+1}{2}}^{+\infty}\Big|\frac{1}{4\ell^2}-\frac{1+O(r|z|)}{(2\ell+1)^2}\Big| \\ &\leqslant \frac{1}{d|z|}\sum_{\ell=\frac{M+1}{2}}^{+\infty}\Big(\frac{4\ell+1}{4\ell^2(2\ell+1)^2}+O\big(\frac{r|z|}{(2\ell+1)^2}\big)\Big) \ll \frac{1}{d|z|M^2}+\frac{r}{dM}.\end{aligned}$$

This leads us to choose M the smallest odd integer $\geqslant n^{1/4+\varepsilon}$. For T_1 we use the fact that $\exp(-mrz)$ is near 1 if M is not too large. Let $J\in\mathbb{N}$ to be specified later:

$$T_1 = \frac{1}{dz}\sum_{m=1}^{M}\frac{(-1)^m}{m^2}(1-mrz+\sum_{j=2}^{J-1}\frac{(-1)^j m^j r^j z^j}{j!}+O(m^J r^J|z|^J)).$$

By the same type of argument as for T_2 (which are standard manipulations on alternating series) we show that

$$\frac{1}{dz}\sum_{m=1}^{M}\frac{(-1)^m}{m^2} = \frac{1}{dz}\sum_{m=1}^{+\infty}\frac{(-1)^m}{m^2}+O\big(\frac{1}{d|z|M^2}\big) = -\frac{\pi^2}{12dz}+O\big(\frac{1}{d|z|M^2}\big),$$

and

$$\frac{r}{d}\sum_{m=1}^{M}\frac{(-1)^m}{m} = \frac{r}{d}\sum_{m=1}^{+\infty}\frac{(-1)^m}{m}+O\big(\frac{r}{dM}\big) = -\frac{r}{d}\log 2+O\big(\frac{r}{dM}\big).$$

Next, for each $2\leqslant j\leqslant J-1$ we have (recall that $r\leqslant d\leqslant n^{1/4-2\varepsilon}$)

$$\frac{r^j|z|^{j-1}}{d}\Big|\sum_{m=1}^{M}(-1)^m m^{j-2}\Big| \ll \frac{r^j|z|^{j-1}M^{j-2}}{d} \ll \frac{n^{-(2+j)\varepsilon}}{d}.$$

Finally for the last error term we have:

$$r^J|z|^{J-1}d^{-1}\sum_{m=1}^{M}m^{J-2} \ll (M|z|)^{J-1}r^J d^{-1} \ll n^{\frac14-J\varepsilon}d^{-1},$$

which is small enough for $J=\lceil\frac{5}{\varepsilon}\rceil$.

This ends the proof of Lemma 9.3.

If we insert (9·8) and Lemma 9.3 in Lemma 9.2 and don't forget (9·1) we obtain Lemma 9.1 in the case $r\neq d$.

10. The term $S_0(0)$, end of the proof of Theorem 1.1

Recall that

$$h(\exp(-z)) = \exp\Big(\frac{\pi^2}{12z} - \frac{1}{2}\log 2 + O(|z|)\Big),$$

if $z \to 0$ with $|\arg z| \leqslant \kappa < \pi/2$. By Lemma 9.1 we have for $|y| \leqslant y_1$:

$$h(\exp(-z))\prod_{r\in\mathcal{D}} U_r(z) = \exp\Big(\frac{\pi^2}{12z}\big(1+\frac{|\mathcal{D}|}{d}\big) + \frac{|\mathcal{D}|}{2}\log z - \frac{\log 2}{2} + \sum_{r\in\mathcal{D}}\big(\frac{r}{d}-1\big)\log 2 + \frac{|\mathcal{D}|}{2}\log\big(\frac{d}{\pi}\big) + O(n^{-\varepsilon})\Big).$$

As in the case $\mathcal{D} = \{1,\dots,d\}$ we insert above the two formulae:

$$\frac{1}{z} = \frac{1}{x_0} - \frac{iy}{x_0^2} - \frac{y^2}{x_0^3} + O\Big(\frac{|y|^3}{x_0^4}\Big) \text{ and } \log z = \log x_0 + O\Big(\frac{|y|}{x_0}\Big),$$

and we apply Lemma 4.1:

$$S_0(0) = \frac{1}{2\pi}\int_{-y_1}^{y_1} \exp(C_{\mathcal{D}} + iy\tilde{B}_{\mathcal{D}} - y^2 A_{\mathcal{D}})\,\mathrm{d}y,$$

with

$$\begin{aligned} C_{\mathcal{D}} = {} & \frac{\pi^2}{12x_0}\big(1+\frac{|\mathcal{D}|}{d}\big) + \frac{|\mathcal{D}|}{2}\log x_0 - \frac{\log 2}{2} + \sum_{r\in\mathcal{D}}\big(\frac{r}{d}-1\big)\log 2 + \frac{|\mathcal{D}|}{2}\log\big(\frac{d}{\pi}\big) \\ & - \frac{C_2|\mathcal{D}|}{dx_0} + \frac{|\mathcal{D}|\log 2}{2} + \sum_{r\in\mathcal{D}}(N_r-k_0)\log 2 - \frac{dx_0}{2}\sum_{r\in\mathcal{D}}(N_r-k_0)^2 \\ & + (n - R_{\mathcal{D}} - Q_{\mathcal{D}})x_0 + O(n^{-\varepsilon}), \end{aligned} \tag{10·1}$$

$$\tilde{B}_{\mathcal{D}} = -\frac{\pi^2}{12x_0^2}\big(1+\frac{|\mathcal{D}|}{d}\big) + \frac{C_2|\mathcal{D}|}{dx_0^2} + \frac{|\mathcal{D}|k_0\log 2}{x_0} - dk_0\sum_{r\in\mathcal{D}}(N_r-k_0) + n - R_{\mathcal{D}} - Q_{\mathcal{D}}, \tag{10·2}$$

$$A_{\mathcal{D}} = \frac{\pi^2}{12x_0^3}\big(1+\frac{|\mathcal{D}|}{d}\big) - \frac{C_2|\mathcal{D}|}{dx_0^3} - \frac{|\mathcal{D}|k_0\log 2}{x_0^2} - \frac{d|\mathcal{D}|k_0^2}{2x_0}. \tag{10·3}$$

Since $k_0 dx_0 = \log 2$, and $C_2 = \frac{\pi^2}{12} - \frac{(\log 2)^2}{2}$, $A_{\mathcal{D}}$, $\tilde{B}_{\mathcal{D}}$ and $C_{\mathcal{D}}$ may be simplified :

$$A_{\mathcal{D}} = \frac{\pi^2}{12x_0^3} - \frac{|\mathcal{D}|(\log 2)^2}{dx_0^3} = \frac{2\sqrt{3}n^{\frac{3}{2}}}{\pi}\Big(1 - \frac{12|\mathcal{D}|(\log 2)^2}{d\pi^2}\Big).$$

Recall that $k_0 = \frac{2\sqrt{3n}\log 2}{\pi d}$ and $|N_r - k_0| \leqslant \frac{n^{\frac14}\sqrt{\log n}}{d^{1/3}|\mathcal{D}|^{2/3}w(n)}$.
By Lemma 2.2 of [6], for $|y| \leqslant y_1$, we have

$$y\tilde{B}_{\mathcal{D}} = -2ydk_0\sum_{r\in\mathcal{D}}(N_r-k_0) + O\Big(n^{-\varepsilon/3}\Big) = yB_{\mathcal{D}} + O\Big(n^{-\varepsilon/3}\Big),$$

say.

We end the computations as in the case $\mathcal{D}=\{1,\dots,d\}$. Finally we obtain

$$\Pi_d^*(n,\mathcal{R}_{\mathcal{D}})=(1+o(1))\frac{\delta}{2\pi}\sqrt{\frac{\pi}{A_{\mathcal{D}}}}\exp\Big(C_{\mathcal{D}}-\frac{B_{\mathcal{D}}^2}{4A_{\mathcal{D}}}\Big). \tag{10·4}$$

To simplify $C_{\mathcal{D}}$ we need some more precise estimations of $R_{\mathcal{D}}$ and $Q_{\mathcal{D}}$:

$$x_0R_{\mathcal{D}}=x_0k_0\sum_{r\in\mathcal{D}}r+O(d^{2/3}|\mathcal{D}|^{1/3}n^{-1/4+\varepsilon}),$$

$$Q_{\mathcal{D}}x_0=\frac{dx_0}{2}\sum_{r\in\mathcal{D}}(N_r-k_0)^2+k_0dx_0\sum_{r\in\mathcal{D}}(N_r-k_0)+\frac{x_0dk_0^2|\mathcal{D}|}{2}-\frac{d|\mathcal{D}|k_0x_0}{2}+O(dn^{-1/4+\varepsilon}).$$

It remains to insert these different formulae in (10·4) to finish the proof of Theorem 1.1.

11. Local stability of $\Pi_d^*(n,\mathcal{R}_{\mathcal{D}})$

In this section we settle a result analogous to Corollary 9.1 of [5]. If $\mathcal{R}_{\mathcal{D}}=\{N_r : r\in\mathcal{D}\}$ and $\mathcal{R}_{\mathcal{D}}^*=\{N_r^* : r\in\mathcal{D}\}$ are two sets of integers satisfying (1·1) and such that N_r^* is near N_r on average then in the estimation of $\Pi_d^*(n,\mathcal{R}_D)$ given by Theorem 1.1, we may replace the N_r by N_r^*. Like in [5] this corollary will be useful for the proofs of the different corollaries announced in the introduction of [6].

Corollary 11.1. *Let $0<\varepsilon<10^{-2}$, $n\geqslant n_0$, $d^3|\mathcal{D}|\leqslant n^{1/2-3\varepsilon}$ and two sets $\mathcal{R}_{\mathcal{D}}=\{N_r : r\in\mathcal{D}\}\in\mathbb{Z}^{|\mathcal{D}|}$, $\mathcal{R}_{\mathcal{D}}^*=\{N_r^* : r\in\mathcal{D}\}\in\mathbb{R}^{|\mathcal{D}|}$ such that*

(i) (1·1) is satisfied for $\mathcal{R}_{\mathcal{D}}$;

(ii) $|N_r-k_0|\leqslant\frac{n^{1/4}\sqrt{\log n}}{d^{1/3}|\mathcal{D}|^{2/3}w(n)}$ for all $r\in\mathcal{D}$ where $w(n)$ is a non decreasing function such that $\lim_{u\to\infty}w(u)=\infty$;

(iii) $\sum_{r\in\mathcal{D}}|N_r-N_r^*|\leqslant\delta+|\mathcal{D}|-1$, $\quad\sum_{r\in\mathcal{D}}|N_r-N_r^*|^2\leqslant\delta^2+|\mathcal{D}|-1$.

Then we have

$$\Pi_d^*(n,\mathcal{R}_D)=q(n)\frac{\delta(1+o(1))}{\sqrt{1-\frac{12|\mathcal{D}|(\log 2)^2}{d\pi^2}}}\Big(\frac{d}{2\sqrt{3n}}\Big)^{|\mathcal{D}|/2}$$
$$\times\exp\Big(-\frac{2\sqrt3(\log 2)^2}{\pi\Big(1-\frac{12|\mathcal{D}|(\log 2)^2}{d\pi^2}\Big)\sqrt n}\Big(\sum_{r\in\mathcal{D}}(N_r^*-k_0)\Big)^2-\frac{\pi d}{2\sqrt{3n}}\sum_{r\in\mathcal{D}}(N_r^*-k_0)^2\Big).$$

Proof. By (iii), $\sum_{r\in\mathcal{D}}|N_r-N_r^*|\leqslant 2d$ and we have

$$\Big|\Big(\sum_{r\in\mathcal{D}}(N_r-k_0)\Big)^2-\Big(\sum_{r\in\mathcal{D}}(N_r^*-k_0)\Big)^2\Big|\leqslant\Big(\sum_{r\in\mathcal{D}}|N_r-N_r^*|\Big)\Big(2\sum_{r\in\mathcal{D}}|N_r-k_0|+\sum_{r\in\mathcal{D}}|N_r-N_r^*|\Big)$$
$$=O\Big(d^{2/3}|\mathcal{D}|^{1/3}\frac{n^{1/4}\sqrt{\log n}}{w(n)}+d^2\Big)=o(n^{1/2}).$$

Similarly we have using also $\delta\leqslant|\mathcal{D}|+1$ (since $\delta\leqslant\min_{a\in\mathcal{D}^c}a$ if $\mathcal{D}^c\neq\varnothing$)

$$\Big|\sum_{r\in\mathcal{D}}(N_r-k_0)^2-\sum_{r\in\mathcal{D}}(N_r^*-k_0)^2\Big|\leqslant\sum_{r\in\mathcal{D}}|N_r-N_r^*|(2|N_r-k_0|+|N_r-N_r^*|)$$
$$\ll(\delta+|\mathcal{D}|-1)|\mathcal{D}|^{-2/3}d^{-1/3}\frac{n^{1/4}\sqrt{\log n}}{w(n)}+\delta^2+|\mathcal{D}|$$
$$\ll|\mathcal{D}|^{1/3}d^{-1/3}\frac{n^{1/4}\sqrt{\log n}}{w(n)}+\delta^2+|\mathcal{D}|=o\Big(\frac{\sqrt n}{d}\Big).$$

This ends the proof of Corollary 11.1.

12. On the normal order of the numbers of parts: proof of Corollary 1.2 of [6]

Let $C_{\mathcal{D}} = \lceil \frac{2\sqrt{3}\log 2}{\pi}\frac{\sqrt{n}}{d^2} - \frac{n^{1/4}\sqrt{\log n}}{d^{4/3}|\mathcal{D}|^{2/3}w(n)}\rceil d$ and $D_{\mathcal{D}} = \lfloor \frac{2\sqrt{3}\log 2}{\pi}\frac{\sqrt{n}}{d^2} + \frac{n^{1/4}\sqrt{\log n}}{d^{4/3}|\mathcal{D}|^{2/3}w(n)}\rfloor d$.
To prove Corollary 1.2 of [6], it is sufficient to show that

$$S^* := \sum_{\substack{N_r\in[C_{\mathcal{D}},D_{\mathcal{D}}[\\ r\in\mathcal{D}\\ \mathcal{R}_{\mathcal{D}}\equiv n\ (\mathrm{mod}\ \delta)}} \Pi_d^*(n,\mathcal{R}_{\mathcal{D}}) = q(n)(1+o(1)). \tag{12·1}$$

As in [5] p. 82, we have to remove the dependence between n and the N_r given by the congruence condition modulo δ. If $1\notin\mathcal{D}$ then $\delta=1$, there are no difficulty and we take $N_r^*=N_r$ for all $r\in\mathcal{D}$. If $1\in\mathcal{D}$ we write $N_1^* = \lfloor\frac{N_1}{\delta}\rfloor\delta$ and $N_r^*=N_r$ for $r\in\mathcal{D}\smallsetminus\{1\}$. Now we will suppose that $1\in\mathcal{D}$, the proof of the other case being similar. Next we apply Corollary 11.1 with these two sets:

$$\begin{aligned} S^* &= (1+o(1))q(n)\frac{\delta}{\sqrt{1-\frac{12|\mathcal{D}|(\log 2)^2}{d\pi^2}}}\Big(\frac{d}{2\sqrt{3n}}\Big)^{|\mathcal{D}|/2} \\ &\times \sum_{\substack{C_{\mathcal{D}}\leqslant N_1\delta< D_{\mathcal{D}}\\ N_r\in[C_{\mathcal{D}},D_{\mathcal{D}}[\\ r\in\mathcal{D}\smallsetminus\{1\}}} \exp\Big(-\frac{2\sqrt{3}(\log 2)^2}{\pi\Big(1-\frac{12|\mathcal{D}|(\log 2)^2}{d\pi^2}\Big)\sqrt{n}}\Big(\delta N_1 - k_0 + \sum_{r\in\mathcal{D}\smallsetminus\{1\}}(N_r-k_0)\Big)^2 \\ &- \frac{\pi d}{2\sqrt{3n}}\Big((\delta N_1-k_0)^2 + \sum_{r\in\mathcal{D}\smallsetminus\{1\}}(N_r-k_0)^2\Big)\Big). \end{aligned}$$

We apply again Corollary 11.1 with $N_i^*=t_i$ so that $|t_i-N_i|\leqslant 1$ for $i\in\mathcal{D}$, we easily see that

$$|\delta t_1' - \delta N_1|^2 + \sum_{r\in\mathcal{D}\smallsetminus\{1\}}(t_r-N_r)^2 \leqslant \delta^2 + |\mathcal{D}| - 1.$$

Thus we have (after replacing $\delta t_1'$ by t_1 in the integral) :

$$\begin{aligned} S^* &= (1+o(1))q(n)\frac{1}{\sqrt{1-\frac{12|\mathcal{D}|(\log 2)^2}{d\pi^2}}}\Big(\frac{d}{2\sqrt{3n}}\Big)^{|\mathcal{D}|/2} \\ &\times \int_{C_{\mathcal{D}}}^{D_{\mathcal{D}}}\cdots\int_{C_{\mathcal{D}}}^{D_{\mathcal{D}}} \exp\Big(-\frac{2\sqrt{3}(\log 2)^2}{\pi\Big(1-\frac{12|\mathcal{D}|(\log 2)^2}{d\pi^2}\Big)\sqrt{n}}\Big(\sum_{r\in\mathcal{D}}(t_r-k_0)\Big)^2 \\ &- \frac{\pi d}{2\sqrt{3n}}\Big(\sum_{r\in\mathcal{D}}(t_r-k_0)^2\Big)\Big)\prod_{r\in\mathcal{D}}\mathrm{d}t_r. \end{aligned}$$

Lemma 12.1. *We have*

$$\begin{aligned} S^* &= (1+o(1))q(n)\frac{1}{\sqrt{1-\frac{12|\mathcal{D}|(\log 2)^2}{d\pi^2}}}\Big(\frac{d}{2\sqrt{3n}}\Big)^{|\mathcal{D}|/2} \\ &\times \int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} \exp\Big(-\frac{2\sqrt{3}(\log 2)^2}{\pi\Big(1-\frac{12|\mathcal{D}|(\log 2)^2}{d\pi^2}\Big)\sqrt{n}}\Big(\sum_{r\in\mathcal{D}}(t_r-k_0)\Big)^2 \\ &- \frac{\pi d}{2\sqrt{3n}}\Big(\sum_{r\in\mathcal{D}}(t_r-k_0)^2\Big)\Big)\prod_{r\in\mathcal{D}}\mathrm{d}t_r \\ &+ O\Big(q(n)\frac{w(n)|\mathcal{D}|^{5/3}}{d^{1/6}\sqrt{\log n}}\exp\Big(-\frac{\pi d^{1/3}\log n}{2\sqrt{3}|\mathcal{D}|^{4/3}w^2(n)}\Big)\Big). \end{aligned}$$

Proof. We have to consider the contribution of terms of type $\int_{D_{\mathcal D}}^{\infty}\int_{C_{\mathcal D}}^{D_{\mathcal D}}\cdots\int_{C_{\mathcal D}}^{D_{\mathcal D}}\cdots$. Clearly it is less than

$$\int_{D_{\mathcal D}}^{\infty}\int_{C_{\mathcal D}}^{D_{\mathcal D}}\cdots\int_{C_{\mathcal D}}^{D_{\mathcal D}}\exp\Big(-\frac{\pi d}{2\sqrt{3n}}\Big(\sum_{r\in\mathcal D}(t_r-k_0)^2\Big)\Big)\prod_{r\in\mathcal D}\mathrm{d}t_r.$$

But

$$\int_{C_{\mathcal D}}^{D_{\mathcal D}}\exp\Big(-\frac{\pi d}{2\sqrt{3n}}(t_r-k_0)^2\Big)\,\mathrm{d}t_r\leqslant\int_{-\infty}^{+\infty}\exp\Big(-\frac{\pi d t_r^2}{2\sqrt{3n}}\Big)\,\mathrm{d}t_r=\sqrt{\frac{2\sqrt{3n}}{d}},$$

and

$$\begin{aligned}\int_{D_{\mathcal D}}^{\infty}\exp\Big(-\frac{\pi d}{2\sqrt{3n}}(t_r-k_0)^2\Big)\,\mathrm{d}t_r&\leqslant\int_{D_{\mathcal D}}^{\infty}\exp\Big(-\frac{\pi d}{2\sqrt{3n}}(D_{\mathcal D}-k_0)(t_r-k_0)\Big)\,\mathrm{d}t_r\\&=\frac{2\sqrt{3n}}{\pi d(D_{\mathcal D}-k_0)}\exp\Big(-\frac{\pi d}{2\sqrt{3n}}(D_{\mathcal D}-k_0)^2\Big).\end{aligned}$$

We have $d^{4/3}|\mathcal D|^{2/3}=d^{8/6}|\mathcal D|^{1/2}|\mathcal D|^{1/6}\leqslant d^{9/6}|\mathcal D|^{1/2}=(d^3|\mathcal D|)^{1/2}\leqslant n^{1/4-\varepsilon}$. Thus the contribution of some term of type $\int_{D_{\mathcal D}}^{\infty}\int_{C_{\mathcal D}}^{D_{\mathcal D}}\cdots\int_{C_{\mathcal D}}^{D_{\mathcal D}}\cdots$ to S^* is

$$O\Big(q(n)\frac{w(n)|\mathcal D|^{2/3}}{d^{1/6}\sqrt{\log n}}\exp\Big(-\frac{\pi d^{1/3}\log n}{2\sqrt3|\mathcal D|^{4/3}w^2(n)}\Big)\Big).$$

Since there are $2|\mathcal D|$ such error terms we obtain Lemma 12.1. It remains to compute the main term to finish the proof of Corollary 1.2 of [6]. We remark that

(12·2)

$$\exp\Big(-\frac{2\sqrt3(\log 2)^2}{\pi\Big(1-\frac{12|\mathcal D|(\log 2)^2}{d\pi^2}\Big)\sqrt n}\Big(\sum_{r\in\mathcal D}(t_r-k_0)\Big)^2-\frac{\pi d}{2\sqrt{3n}}\Big(\sum_{r\in\mathcal D}(t_r-k_0)^2\Big)\Big)=\exp(-\frac12T^tMT),$$

with $T=\begin{pmatrix}t_1-k_0\\t_2-k_0\\\vdots\\t_{\mathcal D}-k_0\end{pmatrix}\in\mathbb R^{|\mathcal D|}$ and M is the symmetric matrix $M=(m_{ij})_{1\leqslant i,j\leqslant|\mathcal D|}$ defined by

$$\frac{m_{ii}}{2}=\frac{2\sqrt3(\log 2)^2}{\pi\Big(1-\frac{12|\mathcal D|(\log 2)^2}{d\pi^2}\Big)\sqrt n}+\frac{\pi d}{2\sqrt{3n}}=:V+U\quad(1\leqslant i\leqslant|\mathcal D|)$$

$$\frac{m_{ij}}{2}=\frac{m_{ji}}{2}=\frac{2\sqrt3(\log 2)^2}{\pi\Big(1-\frac{12|\mathcal D|(\log 2)^2}{d\pi^2}\Big)\sqrt n}=:V\quad(1\leqslant i<j\leqslant|\mathcal D|).$$

A classical result on determinant announces that

$$\det M=2^{|\mathcal D|}U^{|\mathcal D|-1}(U+|\mathcal D|V)=\Big(\frac{\pi d}{\sqrt{3n}}\Big)^{|\mathcal D|}\Big(1-\frac{12|\mathcal D|(\log 2)^2}{d\pi^2}\Big)^{-1}.$$

Thus the function (12·2) is proportional to the density of the law of a Gaussian vector with covariance matrix M^{-1}. We deduce that

$$\int_{-\infty}^{\infty}\!\!\cdots\!\int_{-\infty}^{\infty}\exp\Big(-\frac{2\sqrt3(\log 2)^2}{\pi\Big(1-\frac{12|\mathcal D|(\log 2)^2}{d\pi^2}\Big)\sqrt n}\Big(\sum_{r\in\mathcal D}(t_r-k_0)\Big)^2-\frac{\pi d}{2\sqrt{3n}}\Big(\sum_{r\in\mathcal D}(t_r-k_0)^2\Big)\Big)\prod_{r\in\mathcal D}d_{t_r}$$
$$=(2\pi)^{|\mathcal D|/2}\sqrt{\det M^{-1}}.$$

This ends the proof of Corollary 1.2 of [6].

13. Unequal partitions with equilibrated residue classes: proof of Corollary 1.3 of [6]

For $1 \leqslant a < b \leqslant d$, let $E^*(a,b)$ denote the number of unequal partitions of n such that $N_a = N_b$:

$$E^*(a,b) = \sum_{\substack{N_a=N_b \\ n\equiv aN_a+bN_b \pmod{\delta}}} \Pi_d^*(n,\mathcal{R}_{\{a,b\}}) = \sum_{\substack{N_a=N_b \\ N_a\in[C,D] \\ n\equiv aN_a+bN_b \pmod{\delta}}} \Pi_d^*(n,\mathcal{R}_{\{a,b\}}) + o(q(n)),$$

by Corollary 1.2 of [6] applied with $w(n) = 2^{-2/3}\log\log n$,

$$C = \lceil \frac{2\sqrt{3}\log 2}{\pi}\frac{\sqrt{n}}{d^2} - \frac{n^{1/4}\sqrt{\log n}}{d^{4/3}\log\log n}\rceil d \text{ and } D = \lfloor \frac{2\sqrt{3}\log 2}{\pi}\frac{\sqrt{n}}{d^2} + \frac{n^{1/4}\sqrt{\log n}}{d^{4/3}\log\log n}\rfloor d.$$

Next we apply Theorem 1.1 and Corollary 11.1. Here again we have to remove the condition $n \equiv aN_a + bN_b \pmod{\delta}$ otherwise N_a, N_b, n wouldn't be independent. If $d \geqslant 5$ then $\delta = 1$. For $2 \leqslant d \leqslant 4$, the problematic cases are $(a,b,d) \in \{(1,2,2),\ (1,2,3),\ (1,3,4)\}$. We will handle these cases later. Suppose now that $\delta = 1$.

$$\begin{aligned} E^*(a,b) &= q(n)\frac{1}{\sqrt{1-\frac{24(\log 2)^2}{d\pi^2}}}\Big(\frac{d}{2\sqrt{3n}}\Big) \\ &\quad\times \int_C^D \exp\Big[-\Big(\frac{8\sqrt{3}(\log 2)^2}{\pi\Big(1-\frac{24(\log 2)^2}{d\pi^2}\Big)\sqrt{n}} + \frac{\pi d}{\sqrt{3n}}\Big)(t-k_0)^2\Big]\,\mathrm{d}t + o(q(n)) \\ &= \sqrt{\frac{d}{4\sqrt{3n}}}q(n) + o(q(n)). \end{aligned}$$

As in the proof of Corollary 1.2 of [6] we show that the contributions of $\int_D^\infty \cdots$ and $\int_{-\infty}^C$ are small enough. When $\delta > 1$, as explained in [5] we fixe some congruence conditions on N_a and N_b and next do quite the same computations.

14. Comparison between the number of parts in two residue classes: proof of Corollary 1.4 of [6]

We proceed in the same way as in the previous section. Le $\Delta \in \{0,1\}$. We have to estimate

$$T^*(a,b) = \sum_{\substack{N_a\geqslant N_b+\Delta \\ n\equiv R_{\{a,b\}} \pmod{\delta}}} \Pi_d^*(n,\mathcal{R}_{\{a,b\}}).$$

We take $\Delta = 0$ to examine the unequal partitions with $N_a \geqslant N_b$ and $\Delta = 1$ for the unequal partitions with $N_a > N_b$. In the same way as in the proof of Corollary 1.3 of [6], we obtain

$$\begin{aligned} T^*(a,b) &= q(n)\frac{1}{\sqrt{1-\frac{24(\log 2)^2}{d\pi^2}}}\Big(\frac{d}{2\sqrt{3n}}\Big) \\ &\quad\times \int_{-\infty}^{\infty}\int_{t_b}^{\infty} \exp\Big[-\frac{2\sqrt{3}(\log 2)^2}{\pi\Big(1-\frac{24(\log 2)^2}{d\pi^2}\Big)\sqrt{n}}(t_a+t_b-2k_0)^2 \\ &\quad - \frac{\pi d}{2\sqrt{3n}}((t_a-k_0)^2+(t_b-k_0)^2\Big]\,\mathrm{d}t_a\,\mathrm{d}t_b + o(q(n)). \end{aligned}$$

In the same way we find a similar formula for $T^*(b,a)$. This proves that $T^*(a,b) = q(n)/2 + o(q(n))$.

15. On the d regularity of the unequal partitions: proof of Corollary 1.5 of [6]

Now we suppose that d is fixed in order to use Corollary 1.3 of [6] uniformly. We now study for $\Delta = 0$ or 1 :

$$W^*(a) := \sum_{\substack{N_1,\ldots,N_d \\ n \equiv R \,(\mathrm{mod}\, d) \\ Na \geqslant \Delta + \max_{b \neq a} N_b}} \Pi_d^*(n, \mathcal{R}).$$

We proceed as [5] Sections 12, 13 and like the previous paragraphs of this present paper :

$$W^*(a) = o(q(n)) + \frac{d}{\sqrt{1 - \frac{12(\log 2)^2}{\pi^2}}} \Big(\frac{d}{2\sqrt{3n}}\Big)^{d/2} \int \cdots \int_{t_a \geqslant \max_{b\neq a} t_b} f(t_1, \ldots, t_d)\, \mathrm{d}t_1 \cdots \mathrm{d}t_d,$$

with

$$f(t_1, \ldots, t_d) = \exp\Big\{ - \frac{2\sqrt{3}\log^2 2}{\pi\big(1 - \frac{12(\log 2)^2}{\pi^2}\big)\sqrt{n}} \Big(\sum_{r=1}^{d} (t_r - k_0)\Big)^2 - \frac{\pi d}{2\sqrt{3n}} \sum_{r=1}^{d} (t_r - k_0)^2 \Big\}.$$

By Corollary 1.3 of [6] applied d times (it is why d is fixed), we have

$$\sum_{a=1}^{d} W^*(a) = q(n) + o(q(n)).$$

Since $f(t_1, \ldots, t_d)$ is symmetrical the above terms are asymptotically equal :

$$W^*(a) = \Big(\frac{1}{d} + o(1)\Big) q(n),$$

as it was conjectured in [4] p.334 and in the introduction of [3].

The proof of the second assertion of Corollary 1.5 of [6] is similar. We have only to replace the condition $N_a \geqslant \max_{b\neq a} N_b$ by $N_{\sigma(1)} \geqslant N_{\sigma(2)} \geqslant \cdots \geqslant N_{\sigma(d)}$.

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Cécile Dartyge
Institut Élie Cartan
Université Henri Poincaré–Nancy 1,BP 239
54506 Vandœuvre Cedex, France
dartyge@iecn.u-nancy.fr

Mihály Szalay
Department of Algebra and Number Theory
Eötvös Loránd University
1117 Budapest, Pázmány Péter sétány 1/C
Hungary
mszalay@cs.elte.hu